Math 419 Assignment 2–due Wed. Feb. 9–Final Version

$\mathbf{Z}_+ = \{0, 1, 2, \ldots\}$ $\mathbf{N}$ denotes the set of natural numbers.

1. Let π be a stationary distribution for the transition matrix $\mathbf{P}$, and R_α be a recurrent communicating class such that $\pi(R_\alpha) > 0$. Show that the conditional probability $\pi(\cdot\,|R_\alpha)$ is a stationary distribution for the transition matrix on $\mathbf{R}_\alpha$ given by $(p_{ij})_{i,j\in R_\alpha}$.

2. Let $\{X_n\}$ be simple random walk on $\mathbf{Z}$ with step distribution $P(Y_1 = 1) = p$ and $P(Y_1 = -1) = 1 - p$ for some $0 < p < 1$, $p \neq 1/2$. Prove that X is transient.

3. Consider the Markov Chain with state space $\{1, 2, 3, 4, 5\}$ and transition matrix

$$P = \begin{pmatrix} .5 & 0 & 0 & 0 & .5 \\ 0 & .5 & 0 & .5 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & .25 & .25 & .25 & .25 \\ .5 & 0 & 0 & 0 & .5 \end{pmatrix}.$$

Find the communicating classes and identify which classes are closed, which are recurrent and which are transient.

4. The binary tree is a countable graph in which there is one site in generation 0 which is connected to its two children in generation 1. Each of the 2^n sites in generation n is connected to its parent in generation $n - 1$ and its two children in generation $n + 1$ ($n \in \mathbf{N}$). If S is the set of vertices in the binary graph and for $x \in S$, let $|x| \in \mathbf{Z}_+$ denote the generation of x. Let $X = \{X_n\}$ be a random walk on the binary tree.
(a) Prove X is transient. Hint: Consider $|X_n|$.
(b) Find a stationary measure for X.

In the next 2 questions (and throughout the course) you may use the following slight extension of the Extended Markov Property (proved in class): For bounded or non-negative functions $\phi(X_0, X_1, \ldots)$, for all $m \in \mathbf{Z}_+$, and for each $A \subset S^{m+1}$,

$$\begin{aligned} &E\Big(1((X_0, \ldots, X_m) \in A)\phi(X_m, X_{m+1}, \ldots)\Big) \\ &\quad = E\Big(1((X_0, \ldots, X_m) \in A)E_{X_m}(\phi(X_0, X_1, \ldots))\Big). \end{aligned}$$

5. Consider the Markov chain with state space $\{1, 2, 3, 4\}$ and transition matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ .5 & 0 & .5 & 0 \\ 0 & .5 & 0 & .5 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note that 1 and 4 are absorbing states–once you hit either one you stay there forever. Let $H = \min\{n \geq 0 : X_n \in \{1, 4\}\}$, $H_j = \min\{n \geq 0 : X_n = j\}$ and $h_i = P_i(H_4 < \infty)$.

(a) Find h_j for all j. Hint: Condition on the first step and use the above Markov property to show $h_2 = \frac{h_1}{2} + \frac{h_3}{2}$.
(b) Find $E_2(H)$. Hint: If $X_0 \neq 1$ or 4 then $H(X_0, X_1, \ldots) = 1 + H(X_1, X_2, \ldots)$. Use this to get some equations for $k_j = E_j(H)$ by conditioning on the first step. You may assume those expectations are finite.

6. Assume X is an irreducible recurrent MC. Prove that for all $i, j \in S$,

$$P_i(X_n = j \text{ for some } n \geq 1) = 1.$$