1. Let $X_0$ be initial fortune of gambler. Assume $X_0 = 0$.
(You could assume $X_0 > 0$ if you like, the answers are then modified by subtracting $X_0$).

(a) You bet on $n^{th}$ play if $Y_j = 0$ for $j = 0, \ldots, n-1$ (so you have lost on first $n-1$ plays).

The amount you bet is $2^{n-j-1}$ as the bet has doubled $n-j$ times from an initial bet of $1$.
Your winnings on $n^{th}$ play is $H_n \cdot Y_n$ where $H_n$ is the betting on $n^{th}$ play.

So by above, $H_n = 2^{n-1}$ if $Y_j = 0 \forall j = 0, \ldots, n-1$ and $H_n = 1$ otherwise. Clearly $H_n$ is $\mathcal{F}_n$-measurable, nearly $\mathcal{F}_n$ and so is $H_n$-predictable. Also $0 \leq H_n \leq 2^{n-1}$.

So that $X_n = \sum_{k=1}^{n} H_k Y_k = \frac{n}{2} \sum_{k=1}^{n} 2^{n-k-1} = (M_n - S_n)$. 

(b) $L_T = \min \{ n : Y_n = 1 \}$.

: $H_k = 2^{k-1}$ and $Y_k = 1$ \quad $\forall k \leq T$

: $H_k = 2$ and $Y_k = 1$ \quad $\forall k = T$

: $H_k = 0$ \quad $\forall k > T$.

(i) $T \leq n$. $X_n = \sum_{k=1}^{n} H_k Y_k = \sum_{k=1}^{T} 2^{n-k-1} (1) + 2^{T-1} (1) = 1$ \quad (which you can see directly too).

(ii) $T > n$. $X_n = \sum_{k=1}^{n} H_k Y_k = \sum_{k=1}^{n} 2^{n-k-1} (1) = 1 - 2^n$.

$P(T > n) = P( Y_1 = Y_2 = \ldots = Y_n = 1 ) = 2^{-n}$. 


\[
\sum_{i=0}^{\infty} P(x_n \neq 1) = \sum_{i=1}^{\infty} P(T > n) = \frac{2}{3} - \frac{2}{3}2^{-n} = 0.
\]

By Borel-Cantelli, \( P(x_n \neq 1) \) = 0

\[\Rightarrow x_n = 1 \text{ for large } n \Rightarrow 0.5 \]

\[x_n \to 1 \text{ in } L^1 \]

2. (a) \( P_x (x < y; r > 1) = \int_0^\infty \left( \int_{-\infty}^1 f(x,y) \, dy \right) \, dx = 0 \)

\[= \int_0^1 f(x,y) \, dy = 0 \]

(b) By considering \( g^2 \) separately, we may assume \( g \geq 0 \).

Let \( h(x,y) \) = \( f(x,y) \) \( f(x,1) \) \( f(x,y) \) \( f(x,y) \) \( f(x,y) \)

We need show: \( h(x,y) = E(g(x) | Y) \) \( P \)-a.s.

To see this let \( B \) be a Borel set. It suffices to show.

\[\int_B h(x,y) \, dp = \int_B g(x) \, dp \]

which holds if:

\[\int_B g(x) \, dp = \int_{B_1} g(x) \, f(x,y) \, dx \, dy. \]
By (1), the LHS of (13) is

\[
\int_0^1 \int_0^1 f(x) \, dx \, dy
\]

= \int_0^1 \int_0^1 f(x) \, dx \, dy

= \int_0^1 \int_0^1 f(x) \, dy \, dx

= \int_0^1 \int_0^1 f(x) \, dy \, dx

= \text{RHS of (13)}.

We have also used Fubini's Theorem implicitly in the above.

3. Let \( h(M) = \lim_{n \to \infty} \frac{\nu(\mathcal{E})}{\nu(M)} \).

\[
\int_{\mathcal{E}} 1_{|x_1| > M} \, d\nu = \int_{\mathcal{E}} \frac{1}{h(M)} \sup_{|x_1| > M} \nu(\mathcal{E}) \, d\nu
\]

\[
\sup_{|x_1| > M} \nu(\mathcal{E}) \, d\nu \leq \frac{1}{h(M)} \sup_{|x_1| > M} \nu(\mathcal{E}) \, d\nu
\]

\[
\{x_1\} \text{ are u.i.}
\]

4. \( E(S_n^2) = n \sigma^2 \). \( E(x) \to \infty \) is convex and \( E^n \) is a martingale.

\[
E(S_n^2 - S_{n-1}^2) = E(2S_{n-1}X_n + X_n^2) = 2\sigma^2 + E(X_n^2) = 2\sigma^2
\]

So \( S_n^2 = X_{n-1} + X_n \) is the Doob decomposition where

\[
A_n = \sum_{\nu=1}^{n-1} E(S_{\nu}^2 - S_{\nu-1}^2) \, d\nu = \sum_{\nu=1}^{n-1} \sigma^2 = n \sigma^2
\]

10. \( S_n = S_n^{2} - n\sigma^2 \), \( A_n = n\sigma^2 \).
5. \( X_n \to 0 \), \( X_n \) a.s. converging \( \iff \rightarrow X_\infty \) by HCLT

\( \text{sgn} B(\omega) \to 0 \iff \omega \) so HCLT applies.

\( T = \infty \Rightarrow \mid X_n - x_\infty \mid > \delta \) i.o. \( \Rightarrow \{ x_\infty \} \) is divergent, a set of probability 0.

\( \therefore \text{P}(T = \infty) = 0. \quad \therefore \text{P}(T < \infty) = 1. \)

On \( \Omega \) we have \( X_{n+T} = X_n \quad \forall n \geq T \Rightarrow X_n = X_T \quad \forall n \geq T \)

\( \therefore X_\infty = X_T \) a.s.

\( \mathbb{E}(X_T) = \mathbb{E}(X_\infty) = \mathbb{E}(\lim_{n \to \infty} X_n) = \lim_{n \to \infty} \mathbb{E}(X_n) \) \( (\text{Fclt}) \)

\( \therefore \mathbb{E}(X_\infty) = \mathbb{E}(X_0) \) by supermartingale property.