1. The Original Martingale
Let \( \{Y_k, k \in \mathbb{N}\} \) be iid r.v.’s where \( P(Y_k = 1) = P(Y_k = -1) = \frac{1}{2} \) let \( \mathcal{F}_n = \sigma(Y_1, \ldots, Y_n) \) (\( \mathcal{F}_0 = \{\emptyset, \Omega\} \)), and let \( S_n = \sum_{k=1}^{n} Y_k \). \( Y_k \) is the outcome of the \( k \)th play of a fair game. A gambler uses the following strategy: Bet $1 that the first play is a 1, if you win, quit. If not double your bet on a 1 occurring on the next play. That is bet $2 that the next play is 1. Continue this way until you finally win. You can assume that the gambler can borrow as much money as they need. Let \( X_n \) be the gambler’s fortune after \( n \) plays of the game (of course the gambler may have quit betting well before \( n \) plays).

(a) Show that there is an \((\mathcal{F}_n)\)-predictable process \( H \) where \( 0 \leq H_n \leq K_n \) for some constant \( K_n \) and \( X_n = (H \cdot S)_n \), and conclude that \( X \) is an \((\mathcal{F}_n)\)-martingale.

(b) Show there is an integrable r.v. \( X_\infty \) s.t. \( X_n \to X_\infty \) a.s. and identify \( X_\infty \).

(c) Show that \( L^1 \) convergence fails in (b) above.

2. Let \( X, Y \) be r.v.s with joint density \( f(x, y) \). Recall that the marginal density of \( Y \) is then \( f_Y(y) = \int_{-\infty}^{\infty} f(x, y) \, dx \). Define the conditional density of \( X \) given \( Y = y \) by

\[
 f_{X|Y}(x|y) = \begin{cases} 
 \frac{f(x, y)}{f_Y(y)} & \text{if } f_Y(y) \neq 0, \\
 0 & \text{if } f_Y(y) = 0.
\end{cases}
\]

(a) Show that \( x \to f_{X|Y}(x|y) \) is a probability density for \( P_Y \)-a.a. \( y \).

(b) If \( g : \mathbb{R} \to \mathbb{R} \) is a Borel function with \( g(X) \) integrable, show that \( E(g(X)|Y = y) = \int g(x) f_{X|Y}(x|y) \, dx \) for \( P_Y \) - a.a. \( y \).

3. Let \( \phi \) be a non-negative Borel-measurable function on \( \mathbb{R}_+ \) s.t. \( \lim_{x \to \infty} \frac{\phi(x)}{x} = \infty \). If \( \sup_{i \in I} E(\phi(|X_i|)) < \infty \), show that \( \{X_i : i \in I\} \) is uniformly integrable. Conclude that if for some \( p > 1 \), \( \{X_i : i \in I\} \) is \( L^p \)-bounded then \( \{X_i : i \in I\} \) is uniformly integrable.
4. Assume \( \{Y_k: k \in \mathbb{N}\} \) are iid r.v.'s with mean 0 and finite variance \( \sigma^2 \).
Let \( S_n = \sum_{k=1}^{n} Y_k \) \((S_0 = 0)\) and \( \mathcal{F}_n = \mathcal{F}^Y_n \). Show that \( X_n = S^2_n \) is an \((\mathcal{F}_n)\)-submartingale and find its Doob decomposition. Here you should give an explicit formula for the increasing predictable process in this decomposition and simplify as much as possible.

5. Assume \( \{X_n\} \) is a non-negative supermartingale such that for some real number \( \delta > 0 \), w.p.1 for all \( n \geq 0 \), either \( X_{n+1} = X_n \) or \( |X_{n+1} - X_n| \geq \delta \).
(So \( X_n \) is your fortune after \( n \) plays where there is a minimum bet size.)
Let \( T = \sup\{n : |X_n - X_{n-1}| \geq \delta\} \in \mathbb{Z}_+ \cup \{\infty\} \) be the time of the last bet. Prove that \( P(T < \infty) = 1 \) and \( E(X_T) \leq E(X_0) \). That is, there is a last bet and on average you won't win.