Math 419 Assignment 1–due Wed. Jan. 25

$\mathbf{Z}_+ = \{0, 1, 2, \dots\}$

1. (a) Let $\{X_n : n \in \mathbf{Z}_+\}$ be an S-valued stochastic process and T be an $(\mathcal{F}_n^X)$-stopping time, where S is a countable set. If $B \subset S$ show that

$$S = \min\{n > T : X_n \in B\}$$

is an $(\mathcal{F}_n)$-stopping time. Here $\min \emptyset = \infty$ as usual.
(b) For $i \in S$ and $r \in \mathbf{N}$ inductively define

$$T_i^{(1)} = \min\{n > 0 : X_n = i\},$$

and

$$T_i^{(r+1)} = \min\{n > T_i^{(r)} : X_n = i\}.$$

Show that $T_i^{(r)}$ is an $(\mathcal{F}_n^X)$-stopping time for all natural numbers r.

2. Assume $\{X_n\}$ are independent $\{0, 1\}$-valued random variables such that $P(X_n = 1) = 2^{-n}$. Let

$$L = \sup\{n \in \mathbf{Z}_+ : X_n = 1\}.$$

where $\sup \emptyset = 0$
(a) Prove that $L < \infty$ a. s.
(b) Prove that L is not an $(\mathcal{F}_n^X)$-stopping time.

3. Consider the Markov Chain with state space $\{1, 2, 3\}$ and transition matrix

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1-p & 0 & p \\ 0 & 1 & 0 \end{pmatrix}.$$

Here $p \in [0, 1]$.
(a) Verify that $P^2 = P^4$.
(b) Find P^n for all $n \in \mathbf{N}$.
(c) If $\{X_n\}$ is the Markov Chain with transition matrix P and initial law μ satisfying $\mu(1) = \mu(3) = 1/4$ and $\mu(2) = 1/2$, find $P(X_n = 1)$ for all $n \in \mathbf{Z}_+$.

4. Assume X is an S-valued Markov chain (S countable as usual). Let T be a finite-valued $(\mathcal{F}_n^X)$-stopping time. Show that for all $m \in \mathbf{N}$ and $i, x_k, y_k \in S$,

$$\begin{aligned} &P(X_{T+1} = y_1, \dots, X_{T+m} = y_m | X_k = x_k \text{ for } 0 \le k < T, X_T = i) \\ &\quad = P_i(X_1 = y_1, \dots, X_m = y_m). \end{aligned}$$

This is the Strong Markov Property which says that conditional on the past, the MC starts over again at current location at stopping times as well as constant times.