Math 418/544 Proof of Thm. 3.3

We only need to prove d)⇒e). Assume ∀A ∊ B(S) P(∂A) = 0 implies P_n(A) → P(A).
Let f : S → ℝ be bounded Borel s.t. P(D_f) = 0. We must show

\[ \lim_{n \to \infty} \int f dP_n = \int f dP. \]  (1)

**Lemma.** If f : S → ℝ and A ⊂ ℝ, then

(a) \( f^{-1}(A) \subset D_f \cup f^{-1}(\overline{A}) \)
(b) \( \partial(f^{-1}(A)) \subset D_f \cup f^{-1}(\partial A) \).

**Proof.** (a) Let \( x \in \overline{f^{-1}(A)} \cap D_f^c \). Then there is a sequence \( \{x_n\} \) in \( f^{-1}(A) \) s.t. \( x_n \to x \). As \( x \) is a continuity point of \( f \) we have \( f(x_n) \to f(x) \), and since \( f(x_n) \in A \), it follows that \( f(x) \in \overline{A} \). This proves \( x \in f^{-1}(\overline{A}) \), and (a) follows.

(b) \( \partial f^{-1}(A) = f^{-1}(A) \cap f^{-1}(A^c) = f^{-1}(A) \cap f^{-1}(\overline{A}) \). Now use (a) on each of these sets to conclude that

\( \partial f^{-1}(A) \subset (D_f \cup f^{-1}(\overline{A})) \cap (D_f \cup f^{-1}(\overline{A})) = D_f \cup (f^{-1}(\overline{A} \cap \overline{A})) = D_f \cup f^{-1}(\partial A) \).

\( \square \)

For each natural number \( k \) choose real numbers \( x_1^k < \cdots < x_N^k \) so that

(A) \( \text{Range}(f) \subset [x_1^k, x_N^k] \),  
(B) \( x_{i+1}^k - x_i^k < 2^{-k} \) for all \( i \),  
and (C) \( P(f^{-1}(\{x_i^k\})) = 0 \) for all \( i \).

The latter condition is easy to satisfy since we only need to avoid the countable set of points for which \( P(f^{-1}(\{x\})) > 0 \). Let \( A_i^k = f^{-1}([x_i^k, x_{i+1}^k]) \) for \( i = 1, \ldots, N_k - 1 \) and define

\( f_k = \sum_{i=1}^{N_k-1} x_i^k 1_{A_i^k} \). Then conditions (A) and (B) imply that \( \|f - f_k\|_\infty \leq 2^{-k} \). We also have by (b) of the Lemma and our assumption that \( P(D_f) = 0 \),

\[ P(\partial A_i^k) = P(\partial f^{-1}([x_i^k, x_{i+1}^k])) \leq P(D_f \cup f^{-1}(\{x_i^k, x_{i+1}^k\})) = 0, \]

the last by condition (C). It follows by hypothesis that \( \lim_{n \to \infty} P(A_i^k) = P(A_i^k) \) for all \( i, k \) and hence that for all \( k \),

\[ \lim_{n \to \infty} \int f_k dP_n = \lim_{n \to \infty} \sum_{i=1}^{N_k-1} P_n(A_i^k) = \int f_k dP. \]

The uniform convergence of \( f_k \) to \( f \) therefore shows that for any fixed \( k \),

\[ \limsup_{n \to \infty} \left| \int f dP_n - \int f dP \right| \leq \limsup_{n \to \infty} \left[ \left| \int f - f_k dP_n \right| + \left| \int f_k dP_n - \int f_k dP \right| + \left| \int f_k - f dP \right| \right] \]
\[ \leq 2^{-k} + 0 + 2^{-k} = 2^{1-k}. \]

As \( k \) can be taken arbitrarily large, (1) follows.