Proposition 1.15

Equip \( S = \mathbb{S}^n \) with the metric

\[
d((w, w')) = \sqrt{\sum_{i=1}^{n} (w_i - w'_i)^2}\]

(Note: The only property of \( S^1 \) we use is \( d(w, w) = 2 \) for all \( w \) in \( S^1 \).

\[\text{Ex.}\] If \( (w^n) \) is a sequence in \( S^1 \) such that

\[
\lim_{n \to \infty} d(w^n, w) = 0 \Rightarrow \text{view } \lim_{n \to \infty} w^n = w
\]

Then \( (w^n) \) is a defined and close to \( w \) if the initial\( \Rightarrow \lim_{n \to \infty} w^n = w \).

Lemma 5. \((S, d)\) is compact.

Any sequence \((w^n, n \in \mathbb{N})\) in \( S \) has a convergent subsequence \( w^k \to w \) as \( k \to \infty \).

Proof.\( \text{On } w^n, n \in \mathbb{N}, \text{ so for } n_1 \leq n \text{ and } w^n \text{ is } \mathbb{S}^1 \text{ all } w^k = w \text{ all } k.

Next \( w^k \equiv 0, k \in \mathbb{N} \) so there is a subsequence \( \exists \lim_{n \to \infty} w^k = w \) \(\forall k \),

and \( \exists \lim_{n \to \infty} w^k = w \) \(\forall k \), and \( \exists \lim_{n \to \infty} w^k = w \) \(\forall k \).

Continue in this way to find a subsequence \( \exists \lim_{n \to \infty} w^k = w \) \(\forall k \).

Let \( n_k \equiv n_k \), then \( \exists \lim_{n \to \infty} w^k = w \) \(\forall k \), and so do \(\exists \lim_{n \to \infty} w^k = w \) \(\forall k \).

(2) \(\text{Ex.}\) \( \lim_{n \to \infty} w^n \to w \) in \( S \) where \( \lim_{n \to \infty} w^n = w \). )
Prop. 1.15. Let \( \Omega = \mathbb{R}^N \), \( \mathcal{F} = \text{field of finite-dimensional events} \). Any finitely additive probability \( Q \), on \( \mathcal{F} \), is countably additive onto \( \Omega \) (and so is a probability onto \( \Omega \)).

Proof. Let \( \omega_n \) be a sequence of disjoint sets in \( \mathcal{F} \).

13) \( \bigcup_{n=1}^{\infty} A_n \in \mathcal{F} \)

We must prove: 14) \( Q\left( \bigcup_{n=1}^{\infty} A_n \right) = \sum_{n=1}^{\infty} Q(A_n) \)

Let \( K_N = \bigcup_{n=1}^{N} A_n - \bigcup_{n=1}^{N-1} A_n \downarrow \emptyset \) (this should be clear).

By (3), \( K_N \in \mathcal{F} \) and so \( \exists n, m, w \in W, B_n \subseteq 3^{N, N_n, \alpha} 1.1 \).

13) \( K_N = \{ \omega \in W : (w_1, \ldots, w_m) \in B_n \} \).

We claim (15) \( 1.2 \) if \( N \) is large enough.

15) Suppose not. Choose \( \omega^N = (w_1^N, w_2^N, \ldots) \in K_N \). By (3) and \( \omega \in K_m \) for all \( m < N \) (since \( \omega \) is in \( K_N \) with minimum \( N \) by (1)).

16) \( (w_1^N, \ldots, w_m^N) \in B_m \) for all \( m < N \).

By Lemma C there is a subsequence \( \omega_{k, \omega} \) and \( \omega \in D \) such

\[ w^{N_k} = (w_1^{N_k}, w_2^{N_k}, \ldots) \rightarrow w = (w_1, w_2, \ldots) \quad \text{in} \quad D. \]

Fix \( m \in \mathbb{N} \). By (1) this means

17) for \( k \) large, \( (w_1, \ldots, w_m) = (w_1^{N_k}, \ldots, w_m^{N_k}) \in B_m \) if \( m \leq N_k \).

(In this step by (16) with \( N = N_k \).)
The last condition in (7) \((\forall n)(N_n = n)\) also holds for \(m\) large.

So apply (7) and with \(m\) large to conclude.

\[(w_1, \ldots, w_m) \in Bm \Rightarrow \omega \in k_{m} \quad (\text{by (15)} \text{ with } N = m)\].

As \(m\) is arbitrary \(\omega \in k_m = \omega\), a contradiction.

So our supposition \((\alpha)\) was false and (7) is proved.

Choose \(N_0\), s.t. \(\bigcup_{n=1}^{N_0} A_n = k_{N_0} = \omega\).

\[\therefore (\beta) A_n = \omega \quad \forall n > N_0\]

Now consider (4) (our goal). By (9):

\[Q \left( \bigcup_{n=1}^{N_0} A_n \right) = Q \left( \bigcup_{n=1}^{N_0} A_n \right)\]

\[= \sum_{n=1}^{N_0} Q(A_n) \quad \text{by finite additivity of } Q\text{.}\]

\[= \sum_{n=1}^{N_0} Q(A_n) \quad \text{by (15)}\].

\qed