Proposition 34.16: Let $x, y$ be events, both integrable,

if $x + y$ is defined by $(x, y)$ (i.e., not $\infty - \infty$ or $-\infty - \infty$),

then $x + y$ is integrable and

$\int (x + y) \, d\mu = \int x \, d\mu + \int y \, d\mu$.

Proof: Step 1: $x \geq 0, y \geq 0, a, b \geq 0$.

By part we have

$$\int (ax + by) \, d\mu = \int ax \, d\mu + \int by \, d\mu.$$ 

Step 2. $x = y$, i.e., $\int (x+y) \, d\mu = \int x \, d\mu + \int y \, d\mu$.

All integrals are finite as $(x+y)^2 = x^2 + y^2$ (relatively integrable)

Step 3.

$$\int (x+y)^2 \, d\mu = \int x^2 \, d\mu + \int y^2 \, d\mu.$$ 

Step 4. $x = -y$; i.e., $\int x^2 \, d\mu = \int y^2 \, d\mu$.

Step 5. $a < 0$. Then $\int a \, (x+y)^2 \, d\mu = \int a \, x^2 \, d\mu$.

Step 6. $a > 0$. Then $\int a \, x^2 \, d\mu = \int a \, x^2 \, d\mu$ and the algebra is easier.