Midterm November 15, 2018 Duration: 120 minutes

This test has 7 questions on 12 pages, for a total of 80 points.

- Write your name or student number on every page.
- Attempt to answer all questions for partial credit.
- You may use the back of the previous page, or the blank pages at the end, if you need more room.
- This is a closed-book examination. No aids of any kind are allowed, including: documents, cheat sheets, electronic devices (including calculators, phones, etc.)

First Name: ___________________ Last Name: ___________________

Student No.: ___________________

Signature: ___________________

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Score: 

Student Conduct during Examinations

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.

2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.

3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.

4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.

5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner or invigilator, and may be subject to disciplinary action:

   (i) speaking or communicating with other examination candidates, unless otherwise authorized,

   (ii) purposely exposing written papers to the view of other examination candidates or imaging devices,

   (iii) purposely viewing the written papers of other examination candidates;

   (iv) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,

   (v) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).

6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.

7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.

8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).
1. Short answer.

4 marks  (a) State the Dominated Convergence Theorem.

3 marks  (b) State Chebychev’s inequality.

6 marks  (c) Give an example of a pair of uncorrelated random variables which are not independent. Justify your answer.
2. Let $X_1$ be a discrete r.v. with probability mass function satisfying $p_{X_1}(0) = 3/4$ and $p_{X_1}(1) = 1/4$. Let $X_2$ have a Poisson distribution with mean 2. Let $X_3$ have a uniform distribution on $[0, 2]$. Assume these three random variables are independent. Let $S_3 = X_1 + X_2 + X_3$. Find the following:

3 marks  
(a) The mean of $S_3$.

4 marks  
(b) The variance of $S_3$.

4 marks  
(c) $P(S_3 \leq 1/2)$. 
3. Throughout this question \( \{X_n : n \in \mathbb{N}\} \) and \( X \) are random variables taking values in \([-1,1]\) all defined on a common probability space \((\Omega, \mathcal{F}, P)\). Consider the four different modes of convergence: \( X_n \xrightarrow{P} X \), \( X_n \xrightarrow{a.s.} X \), \( X_n \xrightarrow{w} X \) (weak convergence) and \( X_n \xrightarrow{L^1} X \).

6 marks
(a) There are 12 different possible implications between these four modes. List all of the implications that will hold in this context. (You may use the logical transitivity of “implies” to shorten your answer.) You need not prove them.

4 marks
(b) Prove one of the implications listed in (a). (Choose wisely as some are easier than others!)
(c) Provide a counterexample to illustrate that one of the implications not listed in your answer to (a) is false in general. Justify your example. (Again choose wisely.)
4. True or False. If True provide a proof. If False give (and justify) a counter-example

(a) If $F$ and $G$ are each the distribution function of a random variable, then $H(x) = F(x)G(x)$ is also the distribution function of a random variable.

(b) If $F$ is the probability distribution function of a non-negative random variable with mean one then

$$f(x) = \begin{cases} 
1 - F(x) & \text{if } x \geq 0 \\
0 & \text{if } x < 0 
\end{cases}$$

is the probability density function of a random variable.
5. Assume \( \{X_n : n \in \mathbb{N}\} \) are i.i.d. random variables such that \( E(X_1) = +\infty \). If 
\[ S_n = \sum_{i=1}^{n} X_i, \]
prove that \( \lim_{n \to \infty} \frac{S_n}{n} = \infty \) a.s.
6. Let $X$ be an integrable random variable such that $E(X^4) \leq E(X)^4$. Prove that $X$ is a.s. constant.
7. Let $X = \{X_n : n \in \mathbb{N}\}$ be a sequence of random variables. The $n$th empirical distribution function associated with $X$ is $F_n(x, \omega) = \frac{1}{n} \sum_{k=1}^{n} 1(X_k(\omega) \leq x)$.

(a) If $\{X_n : n \in \mathbb{N}\}$ are independent, prove that for each fixed $x \in \mathbb{R}$, there is a constant $G(x) \in [0, 1]$ such that
$$\limsup_{n \to \infty} F_n(x, \omega) = G(x) \quad a.s.$$

(b) If $\{X_n : n \in \mathbb{N}\}$ are i.i.d. random variables with distribution function $F(x)$, prove that for each fixed real number $x$,
$$\lim_{n \to \infty} F_n(x, \omega) = F(x) \quad a.s.$$
(c) If \( \{X_n : n \in \mathbb{N}\} \) are as in (b), prove that

\[
\lim_{n \to \infty} F_n(x, \omega) = F(x) \quad \text{for all } x \in \mathbb{R} \text{ a.s.,}
\]

that is, the result in (b) holds simultaneously for all real \( x \) on a set of probability one.
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