Monotone Convergence Theorem 1.17

If \( x_{n \downarrow} \) are e.v.s s.t. \( x_{n \downarrow} \leq x \) for all \( n \), then \( \int x \, d\mu \geq \lim \int x_n \, d\mu \).

Proof: We saw in class that \( \int x_n \, d\mu \leq \int x \, d\mu \).

Consider the reverse inequality. Let \( 0 < b \leq 1 \). Fix \( k \in \mathbb{N} \).

Recall \( x^{(k)} \) is the \( k \)-th discrete skeleton of \( x \).

Let \( B_n = \{ \omega : x_n(\omega) \geq b \} \cap B_n \) be an \( \mathcal{F}_n \)-measurable set. \( B_n \) is also \( \mathcal{B} \)-measurable.

Claim 1: \( B_n \) is \( \mathcal{B} \)-measurable, i.e., \( \mu(B_n) = 2 \).

Case 1: \( x(\omega) = 0 \). Then \( x^{(k)}(\omega) = 0 \) so \( \omega \in B_n \) for all \( n \).

Case 2: \( x(\omega) > 0 \). Then \( x_n(\omega) \geq b \) for \( n \) large enough, \( x^{(k)}(\omega) \geq b \) so \( \omega \in B_n \).

So \( \mu(B_n) = 2 \) for all \( n \).

Lemma (1.11): \( \int B_n \, d\mu = \int x^{(k)} \, d\mu \)

Since \( 1_{B_n} x_n \geq 1_{B_n} b x^{(k)} \),

\[
\int 1_{B_n} x_n \, d\mu \geq \int 1_{B_n} b x^{(k)} \, d\mu = b \int 1_{B_n} x^{(k)} \, d\mu
\]

where \( x^{(k)} = \frac{1}{k} \sum x_i \).

\[
\Rightarrow 0 = \lim_{n \to \infty} \mu(A_n \cap B_n) = \mu(A_n)
\]

(\( \mu \) is countably increasing sequence of sets)

\[
\Rightarrow L = \lim_{n \to \infty} \int x_n \, d\mu = b \sum_{i=1}^{\infty} x_i \mu(A_i) = b \int x^{(k)} \, d\mu
\]

Now let \( b \uparrow 1 \) and then \( k \to \infty \) to see \( L = \int x \, d\mu \).

\( \Box + \Box \Rightarrow L = \int x \, d\mu \). \( \square \)