Lemma 1.10 \[ P\left( \left| \frac{S_n}{n} - \frac{1}{2} \right| \geq \frac{1}{3} \right) \leq M^2/4n. \]

Notation. \( I_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} \)

\( I_A \) is the indicator of the event \( A \).

Proof. \[ P\left( \left| \frac{S_n}{n} - \frac{1}{2} \right| \geq \frac{1}{3} \right) \]

\[ = \sum_{\omega \in \Omega} \frac{1}{2^n} \left( \frac{\omega}{n} - \frac{1}{2} \right)^2 \cdot 2^{-n} \]

\[ = \sum_{\omega \in \Omega} \frac{2^n}{2^{3n}} \left[ \frac{S_n}{n} - \frac{1}{2} \right]^2 \cdot \frac{M^2}{2^2} \]

\[ = \frac{M^2}{2^2} \sum_{\omega \in \Omega} \left[ \frac{S_n}{n} - \frac{1}{2} \right]^2 \cdot \frac{2^n}{2^{3n}} \]

Let \( Y_{2n}(\omega) = X_{2n}(\omega) - \frac{1}{2} = \begin{cases} \frac{1}{2} & \text{if } X_{2n}(\omega) > 0 \\ -\frac{1}{2} & \text{if } X_{2n}(\omega) > 0 \end{cases} \). Go by above.

\[ P\left( \left| \frac{S_n}{n} - \frac{1}{2} \right| \geq \frac{1}{3} \right) = M^2/4n \]

\[ = \frac{M^2}{4n} \sum_{\omega \in \Omega} \left( \frac{1}{2} Y_{2n}(\omega) \right)^2 \]

\[ = \frac{M^2}{4n} \sum_{\omega \in \Omega} \frac{1}{4} Y_{2n}(\omega)^2 + \frac{1}{4} \sum_{1 \leq j < k \leq n} \frac{1}{4} Y_{2n}(\omega) Y_{2n}(\omega) \]

\[ = \frac{M^2}{4n} \left[ \frac{1}{4} \sum_{\omega \in \Omega} Y_{2n}(\omega)^2 + \frac{1}{4} \sum_{1 \leq j < k \leq n} \frac{1}{4} Y_{2n}(\omega) Y_{2n}(\omega) \right] \]

\[ = \frac{M^2}{4n} \left[ \frac{1}{4} \sum_{\omega \in \Omega} \frac{n}{n} + \frac{M^2}{4n} \sum_{1 \leq j < k \leq n} \left[ \frac{1}{2} 2^{-n} - \frac{1}{2} 2^{n-1} \right] \right] \]

\[ = M^2/4n \]