1. Let $X$ be a r.v. whose characteristic function $\phi_X$ satisfies $\int_{-\infty}^{\infty} |\phi_X(t)| \, dt < \infty$. Prove that $X$ has a p.d.f. given by

$$f_X(x) = \int_{-\infty}^{\infty} (2\pi)^{-1/2} e^{-itx} \phi_X(t) \, dt.$$ 

2. This question demonstrates that continuity at zero of the limit of the characteristic functions is essential in Levy’s continuity theorem.

Let $X_n$ have a uniform distribution on $[-n,n]$, for $n \in \mathbb{N}$.

(a) Prove that the characteristic functions $\{\phi_{X_n}\}$ converge pointwise to a function $\phi$ which is discontinuous at $t = 0$ and find $\phi$.

(b) Prove that $X_n$ does not converge weakly to any random variable.

3. Let $P_n$ converge weakly to $P$ in $M_1(\mathbb{R})$. Let $F_n$ and $F$ be the distribution functions corresponding to $P_n$ and $P$, respectively. If $P$ is atomless (i.e. $P(\{x\}) = 0$ for all $x$), prove that

$$\lim_{n \to \infty} \sup_{x \in \mathbb{R}} |F_n(x) - F(x)| = 0.$$ 

4. Assume $\{Y_n, n \in \mathbb{N}\}$ are iid random variables with density function $f(x) = e^{-x}1(x \geq 0)$. Let $X_n = \max(Y_1,Y_2,\ldots,Y_n)$. Prove that $\lim_{n \to \infty} \frac{X_n}{\log n} = 1$ a.s. You may use Q8 on HW 5 if it helps.