Math 418/544 Assignment 5 Due Wed. Nov. 14

1. Let \( \{ X_n \} \) be a sequence of uncorrelated r.v.'s with common \( \mu \) such that \( \sup_n \text{Var}(X_n) < \infty \). If \( S_n = \sum_{k=1}^{n} X_k \), show that \( n^{-2} \sum_{k=1}^{n} S_k \) converges in probability as \( n \to \infty \) and identify the limit.

2. In this question you should do all 3 parts but only hand in (c).
   (a) Show that \( d(X, Y) = E(|X - Y| \wedge 1) \) defines a metric on the space of r.v.'s on a given probability space, where we identify r.v.'s \( X \) and \( Y \) if \( X = Y \) a.s.
   (b) Show that \( d(X_n, X) \to 0 \) iff \( X_n \to X \) in probability.
   (c) Show that the metric \( d \) is complete. That is if \( d(X_m, X_n) \to 0 \) as \( m, n \to \infty \), prove that there is a r.v. \( X \) so that \( d(X, X_n) \to 0 \) as \( n \to \infty \).

3. Let \( f : [0, 1] \to \mathbb{R} \) be an \( \alpha \)-Hölder function for some \( \alpha \in (0, 1] \), that is, for some constant \( L > 0 \), \( |f(x) - f(y)| \leq L|x - y|^{\alpha} \) for all \( x, y \in [0, 1] \). If \( p_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^{k}(1-x)^{n-k} f(k/n) \) is the \( n \)th Bernstein polynomial of \( f \), prove that \( \|f - p_n\|_{\infty} \leq C n^{-\alpha/2} \), where \( C \) depends only on \( L \) and \( \alpha \).

4. Let \( \{ X_n \} \) be iid exponential r.v.'s with rate \( \lambda > 0 \). Find the density of \( X_1 + \cdots + X_n \) by inductively calculating the \( n \)-fold convolution of \( f \) with itself where \( f \) is the density of \( X_1 \).

5. A sequence of reals \( \{ x_j \} \) in \( [0, 1] \) is said to be uniformly distributed iff for every \( a < b \) in \( [0, 1] \),
   \[
   \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} 1_{(a,b)}(x_j) = b - a.
   \]
   Prove that such a sequence exists.
   **Hint.** Show that it suffices to show the above result holds for rational values of \( a \) and \( b \).

6. Practice Questions (not to hand in):
   (a) p. 72 #2.3.13, p. 77 #2.4.3
   (b) Assume \( Y \) is a non-negative r.v. such that \( E(Y^2) < \infty \). Prove that
   \[
   P(Y > 0) \geq E(Y^2)/E(Y^2).
   \]
   **Hint:** One approach is to apply Hölder’s inequality \( p = q = 1/2 \) to \( Y1(Y > 0) \).
(c) Let $X_1, \ldots, X_n$ be random variables such that the distribution function of $X = (X_1, \ldots, X_n)$ can be factored as $F_X(x_1, \ldots, x_n) = \prod_{i=1}^{n} G_i(x_i)$ for some non-negative Borel functions $G_i$. Does this imply that $X_1, \ldots, X_n$ are independent r.v.s? Prove or provide a counter-example.