Math 418/544 Assignment 4 Due Wed. Nov. 9 at start of class

1. Let \( \{X_n\} \) be a sequence of uncorrelated r.v.’s with common \( \mu \) such that \( \sup_n \text{Var}(X_n) < \infty \). If \( S_n = \sum_{k=1}^{n} X_k \), show that \( n^{-2} \sum_{k=1}^{n} S_k \) converges in probability as \( n \to \infty \) and identify the limit.

2. (a) Show that \( d(X, Y) = E(|X - Y| \wedge 1) \) defines a metric on the space of r.v.’s on a given probability space, where we identify r.v.’s \( X \) and \( Y \) if \( X = Y \) a.s.

(b) Show that \( d(X_n, X) \to 0 \) iff \( X_n \to X \) in probability.

(c) Show that the metric \( d \) is complete. That is if \( d(X_m, X_n) \to 0 \) as \( m,n \to \infty \), prove that there is a r.v. \( X \) so that \( d(X_n, X) \to 0 \) as \( n \to \infty \).

3. Let \( f : [0, 1] \to \mathbb{R} \) be an \( \alpha \)-Hölder function for some \( \alpha \in (0, 1] \), that is, for some constant \( L > 0 \), \( |f(x) - f(y)| \leq L|x - y|^\alpha \) for all \( x, y \in [0, 1] \). If \( p_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f(k/n) \) is the \( n \)th Bernstein polynomial of \( f \), prove that \( \|f - p_n\|_\infty \leq C n^{-\alpha/2} \), where \( C \) depends only on \( L \) and \( \alpha \).

4. Let \( \{X_n\} \) be iid exponential r.v.’s with rate \( \lambda > 0 \). Find the density of \( X_1 + \cdots + X_n \) by inductively calculating the \( n \)-fold convolution of \( f \) with itself where \( f \) is the density of \( X_1 \).

5. A sequence of reals \( \{x_j\} \) in \( [0, 1] \) is said to be uniformly distributed iff for every \( a < b \) in \( [0, 1] \),

\[
\lim_{n \to \infty} \left[ \frac{\sum_{j=1}^{n} \mathbb{1}_{(a, b]}(x_j)}{n} \right] = b - a.
\]

Prove that such a sequence exists.

**Hint.** Show that it suffices to show the above result holds for rational values of \( a \) and \( b \).

6. Practice Questions (not to hand in):

   (a) p. 72 #2.3.13, p. 77 #2.4.3

   (b) Assume \( Y \) is a non-negative r.v. such that \( E(Y^2) < \infty \). Prove that

\[
P(Y > 0) \geq E(Y^2)/E(Y^2).
\]

**Hint:** One approach is to apply Hölder’s inequality \((p = q = 1/2)\) to \( Y^1(Y > 0) \).

(c) (Extension of the Second Borel-Cantelli Lemma) Assume \( \{A_n\} \) are events such that \( \sum_{n=1}^{\infty} P(A_n) = \infty \). Let

\[
\alpha = \lim \sup_{n \to \infty} \frac{(\sum_{j=1}^{n} P(A_j))^2}{\sum_{j,k=1}^{n} P(A_j \cap A_k)}.
\]

i. Show that \( \alpha \leq 1 \).

ii. Use (b) to show that \( P(A_n \text{ i.o.}) \geq \alpha \).

iii. Show that \( \alpha = 1 \) if \( A_j \) and \( A_k \) are independent events whenever \( |j - k| > N \) for some non-negative integer \( N \), hence extending the second Borel Cantelli Lemma.