Math 418/544 Assignment 3 Due Wed. Oct. 26 at start of class

1. Let \( X \) be a non-negative r.v. with mean 0. Use Markov’s inequality to show that \( X = 0 \) a.s.

2. If \( Y \) is a non-negative r.v. show that for any \( p > 0 \), \( E(Y^p) = \int_0^\infty px^{p-1}P(Y \geq x)dx \).
   \textbf{Hint:} \( y^p = \int_y^\infty px^{p-1}dx \).

3. (a) For a fixed \( x > 0 \) show there is a non-constant r.v. so that \( P(|X - \mu| \geq x) = \sigma^2/x^2 \), i.e., equality holds in Chebychev’s inequality. (Here \( \mu \) and \( \sigma^2 \) are the mean and variance of \( X \).)
   (b) Show there is no non-constant r.v. with finite mean so that equality holds in Chebychev’s inequality for all \( x > 0 \).

4. Use Jensen’s inequality to show that for any r.v., \( X \), on \( (\Omega, \mathcal{F}, P) \), the function \( p \to \|X\|_p = \infty \) is a monotone increasing function on \( (0, \infty) \). Here \( \|X\|_p = \left[ \int |X|^p dP \right]^{1/p} \).

5. Let \( X, Y \) and \( Z \) be independent r.v.’s with a uniform distribution on \([0, 2]\). Find:
   (a) the joint p.d.f. of \((X, Y)\) (b) \( P(X + Y \leq 1) \) (c) \( P(X + Y \leq Z) \)
   (d) the p.d.f of \( X/Y \).

6. Give an example of events \( A_1, A_2 \) and \( A_3 \) on a probability space \((\Omega, \mathcal{F}, P)\) such that \( 0 < P(A_i) < 1 \) for all \( i \), \( P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3) \), but \( A_1 \) and \( A_2 \) are not independent events.

7. Let \( X_1, \ldots, X_n \) be random variables such that the distribution function of \( X = (X_1, \ldots, X_n) \) can be factored as \( F_X(x_1, \ldots, x_n) = \prod_{i=1}^n G_i(x_i) \) for some non-negative Borel functions \( G_i \). Does this imply that \( X_1, \ldots, X_n \) are independent r.v.s? Prove or provide a counter-example.
   \textbf{Hint:} Consider \( n = 2 \) first.

8. Practice Questions (not to hand in): p.34 #1.6.6, p. 45 #2.1.5

Hölder’s inequality states that if \((\Omega, \mathcal{F}, \mu)\) is a measure space, \( \infty > p, q > 1 \) satisfy \( p^{-1} + q^{-1} = 1 \), and \( |f|^p \) and \( |g|^q \) are integrable, then \( fg \) is \( \mu \)-integrable and
\[
\left| \int fg \, d\mu \right| \leq \left[ \int |f|^p \, d\mu \right]^{1/p} \left[ \int |g|^q \, d\mu \right]^{1/q} \equiv \|f\|_p \|g\|_q.
\]
Prove this using Jensen’s inequality. One approach is to consider the probability \( P(A) = \int 1_A |f|^p \, d\mu [\int |f|^p \, d\mu]^{-1} \) (we may assume wlog that \( \int |f|^p \, d\mu > 0 \)–why is this?) and write \( \int |f||g|d\mu \) as an integral w.r.t. \( P \). (You may assume \( P \) is a probability, as this follows from our proof of Lemma 1.23.)