1. Assume that the two-dimensional random vector \( X = (X_1, X_2) \) has joint
p.d.f. \( f(x_1, x_2) \). That is, \( f \) is a non-negative Borel function s.t. \( \int f dy = 1 \),
and for any \( x \in \mathbb{R}^2 \), \( P(X \leq x) = \int 1_{(-\infty,x]}(y)f(y)dy \),
where \( dy \) denotes integration w.r.t. \( m_2 \), two-dimensional Lebesgue measure.

(a) Prove that for any Borel set \( B \) in \( \mathcal{B}(\mathbb{R}^2) \), \( P(X \in B) = \int 1_B(y)f(y)dy \).

(b) Assume that \( X \) is uniformly distributed on \([0, 2]^2\), that is,
\[
f(x) = \begin{cases} 
\frac{1}{4} & \text{for } x \in [0, 2] \times [0, 2] \\
0 & \text{otherwise.}
\end{cases}
\]
Find the d.f. and p.d.f. of \( Y = X_1 + X_2 \).

2. A searchlight is distance 1 from a wall. Let \( Q \) denote the point on the wall
directly opposite it and assume it scans along the wall so that at any given
time, the angle, \( \theta \), the beam of light makes with the perpendicular from
the light to \( Q \) is uniform on \((-\pi/2, \pi/2)\). Let \( X \in \mathbb{R} \) be the position of the beam
on the wall as measured from \( Q \). Find the distribution and density functions
for \( X \). Being mathematicians, we are assuming the wall is infinite in length.

3. Let \( \{X_n : n \in \mathbb{N}\} \) be a sequence of random variables and \( N \) be an \( \mathbb{N} \)-valued
random variable, all on the same probability space \( (\Omega, \mathcal{F}, P) \). Prove that \( X_N \)
is a random variable. (You should define this mapping on \( \Omega \) carefully.)

4. Let \( (\Omega, \mathcal{F}, P) \) be the infinite coin-tossing space we constructed in class. That
is, \( \Omega = \{0, 1\}^\mathbb{N} \), \( \mathcal{F} \) is the sigma-field generated by the finite-dimensional sets,
and \( P \) is the coin-tossing probability. Recall that \( X_n(\omega) = \omega_n \in \{0, 1\} \)
is the result of the \( n \)th toss. Define \( X(\omega) = \sum_{n=1}^{\infty} X_n(\omega)2^{-n} \).

(a) Show that the above series converges for every \( \omega \in \Omega \) and defines a r.v.
\( X \) taking values in \([0, 1]\).

(b) Show that \( X \) has a uniform distribution on \([0, 1]\).

**Hint:** One approach is to first find \( P(X \in [i2^{-m}, (i+1)2^{-m})] \) for
\( i = 0, \ldots, 2^m - 1 \).

Note that this gives another construction of Lebesgue measure on the
unit interval \([0, 1]\). One could also proceed in the opposite direction and
construct the coin-tossing probability from Lebesgue measure on \([0, 1]\).

5. Practice Questions (not to hand in): p. 16 # 1.3.8, p 21 # 1.4.1,
p. 26 # 1.5.4, 1.5.6