1. Let $\mathcal{F}_0$ be the field of finite disjoint unions of right semiclosed intervals on $\mathbb{R}$. (Recall this was denoted by $\mathcal{U}_1$ in class.) Define $\mu : \mathcal{F}_0 \to [0, 1]$ by

$$
\mu(A) = \begin{cases} 
1 & \text{if } [x, \infty) \subset A \text{ for some } x > 0, \\
0 & \text{otherwise.}
\end{cases}
$$

Prove that $\mu$ is a finitely additive probability on $(\mathbb{R}, \mathcal{F}_0)$ but is not a probability measure on $(\mathbb{R}, \mathcal{F}_0)$.

2. Let $F : \mathbb{R}^d \to [0, 1]$ be a $d$-dimensional distribution function, i.e.,

(i) For any $a \leq b$ in $\mathbb{R}^d$, $\Delta_{(a,b]} F = F(x_1, \ldots, x_d) |_{x_1=a_1}^{b_1} \cdots |_{x_d=a_d}^{b_d} \geq 0.$ (Or see p. 14 of the text for an equivalent definition of $\Delta_{(a,b]} F$.)

(ii) $\lim_{z \downarrow x} F(z) = F(x)$ for all $x$ (this means $z_i > x_i$ in taking the limit).

(iii) For any $j = 1, \ldots, d$, and any $(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_d) \in \mathbb{R}^{d-1}$,

$$
\lim_{x_j \downarrow -\infty} F(x_1, x_2, \ldots, x_d) = 0,
$$

and $\lim_{x \to \infty} F(x) = 1$, where $x \to \infty$ means each coordinate $x_j \to \infty$.

Prove that $F(x_1, \ldots, x_d)$ is non-decreasing in each variable $x_i$.

3. Give an example of a function $F : \mathbb{R}^2 \to [0, 1]$ such that ((b) and (c) below are just the 2-dimensional versions of (ii) and (iii), respectively)

(a) $F(x, y)$ is non-decreasing in $x$ and in $y$.

(b) $\lim_{(x', y') \downarrow (x, y)} F(x', y') = F(x, y)$.

(c) $\lim_{x \to \infty, y \to \infty} F(x, y) = 1$, $\lim_{x \downarrow -\infty} F(x, y) = 0$ for all $y$, and $\lim_{y \downarrow -\infty} F(x, y) = 0$ for all $x$,

but $F$ is not a two-dimensional distribution function. This, together with Q2, shows (a)-(c) are strictly weaker than (i)-(iii) in general.

**Read Section 1.2.**

4. If $F, G$ are distribution functions on the line, we say $F$ is stochastically smaller than $G$ and write $F \prec G$ iff $G(x) \leq F(x)$ for all $x \in \mathbb{R}$ (yes it is correct!).

(a) Suppose $X, Y$ are random variables on $(\Omega, \mathcal{F}, P)$ such that $P(X \leq Y) = 1$. Prove that $F_X \prec F_Y$. Here $F_X(x) = P(X \leq x)$ is the distribution function of $X$.

(b) Suppose $F \prec G$ ($F$ and $G$ are distribution functions). Show there are random variables $X, Y$ on some probability space $(\Omega, \mathcal{F}, P)$ such that $F_X = F$, $F_Y = G$ and $P(X \leq Y) = 1$.

5. Practice Questions (not to hand in): p.12-13 #1.2.4, #1.2.5, #1.2.7; p. 16 #1.3.5, #1.3.6