1. Prove Lemma 0.6. That is, show that if \( C = \{ \omega \in \Omega : \lim_n S_n(\omega)/n = 1/2 \} \) and \( \hat{C} = \{ \omega \in \Omega : \lim_m S_m^2(\omega)/m^2 = 1/2 \} \), then \( \hat{C} = C \). Recall that \( \Omega = \{0,1\}^N \) and \( S_n(\omega) = \sum_{k=1}^n \omega_k \).

2. (a) If \( \{F_n : n \in \mathbb{N}\} \) is an increasing sequence of fields of a set \( \Omega \), prove that \( \bigcup_{n=1}^\infty F_n \) is also a field.

(b) Show by counterexample, say with \( \Omega = \mathbb{N} \), that the result in (a) is false if “field” is replaced everywhere by “\( \sigma \)-field”.

3. (a) If \( \Omega = \{1,2,3,4\} \) and \( \mathcal{F} = 2^{\{1,2,3,4\}} \), give an example of two distinct probabilities \( P_1 \) and \( P_2 \) on \( (\Omega, \mathcal{F}) \) that agree on a collection of sets \( \mathcal{C} \) satisfying \( \sigma(\mathcal{C}) = \mathcal{F} \).

(b) Let \( (\Omega, \mathcal{F}) \) be a measurable space and \( \mathcal{C} \subset \mathcal{F} \) be closed under finite unions and satisfy \( \sigma(\mathcal{C}) = \mathcal{F} \). Prove that if two probabilities agree on \( \mathcal{C} \), then they agree on \( \mathcal{F} \).

(c) Does the result in (b) continue to hold if \( \mathcal{C} \) is closed under complementation rather than finite unions? Prove your answer or provide a counter-example.

4. Read Section 1.2 in the text.

If \( F, G \) are distribution functions on the line, we say \( F \) is stochastically smaller than \( G \) and write \( F \prec G \) iff \( G(x) \leq F(x) \) for all \( x \in \mathbb{R} \) (yes it is correct!).

(a) Suppose \( X, Y \) are random variables on \( (\Omega, \mathcal{F}, P) \) such that \( P(X \leq Y) = 1 \). Prove that \( F_X \prec F_Y \). Here \( F_X(x) = P(X \leq x) \) is the distribution function of \( X \).

(b) Suppose \( F \prec G \) (\( F \) and \( G \) are distribution functions). Show there are random variables \( X, Y \) on some probability space \( (\Omega, \mathcal{F}, P) \) such that \( F_X = F, F_Y = G \) and \( P(X \leq Y) = 1 \).