Some Notes on Subsequences and Limsup

We start with the completion of the proof of

**Theorem 5.4.** If \( x_n \) is a real-valued sequence, then \( \liminf_{n \to \infty} x_n = \min L(\{x_N\}) \) and \( \limsup_{n \to \infty} x_n = \max L(\{x_N\}) \).

**Proof** We were considering \( \bar{L} = \limsup x_n \) and had reduced the result to showing that \( \limsup_{n \to \infty} x_n \in L(\{x_N\}) \). If \( \bar{L} = \pm \infty \) we also had completed the proof. So consider \( \bar{L} \) real-valued which was left as an exercise in class.

First we show that

(1) \( \forall \varepsilon > 0 \ \exists N \) there is a natural number \( n > N \) so that \( |x_n - \bar{L}| < \varepsilon \).

Since \( \bar{x}_n \downarrow \bar{L} \), there is a natural number \( N_1 \) such that \( n \geq N_1 \) implies \( |\bar{x}_n - \bar{L}| < \varepsilon /2 \). We may assume wlog that \( N_1 > N \). Since \( \bar{x}_{N_1} = \sup \{x_n : n \geq N_1\} \), there is a natural number \( n \geq N_1 > N \) so that \( x_n > \bar{x}_{N_1} - \varepsilon /2 \) (otherwise \( \bar{x}_{N_1} - \varepsilon /2 \) would be an upper bound for \( \{x_n : n \geq N_1\} \)). Therefore \( n > N \) and

\[
|x_n - \bar{L}| \leq |x_n - \bar{x}_{N_1}| + |\bar{x}_{N_1} - \bar{L}| < \bar{x}_{N_1} - x_n + \varepsilon /2 < \varepsilon.
\]

This proves (1).

Now inductively define \( n_1 < n_2 < \ldots < n_k < \ldots \) so that \( |x_{n_k} - \bar{L}| < 1/k \) for all \( k \). By (1) with \( N = 1 \) and \( \varepsilon = 1 \) we may pick \( n_1 \) so that \( |x_{n_1} - \bar{L}| < 1 \). Given \( n_1 < \ldots < n_k \) satisfying the above, apply (1) with \( N = n_k \) and \( \varepsilon = 1/(k+1) \) to find \( n_{k+1} > n_k \) so that \( |x_{n_{k+1}} - \bar{L}| < 1/(k+1) \). This completes the inductive construction of \( \{x_{n_k}\} \). Clearly \( x_{n_k} \to \bar{L} \), proving \( \bar{L} \in L(\{x_n\}) \), as required.

**Example 1.** Let \( Q \cap (1,2) = \{r_n : n \geq 1\} \) where \( r_m \neq r_n \) if \( m \neq n \). Claim \( L(\{r_n\}) = [1,2] \).

If \( x \in L(\{r_n\}) \), then there are \( r_{n_k} \) so that \( r_{n_k} \to x \). As \( r_{n_k} \in (1,2) \), this implies \( x \in (1,2) = [1,2] \) and so \( L(\{r_n\}) \subset [1,2] \).

Next fix \( x \in [1,2] \). To show \( x \in L(\{r_n\}) \) we need to find \( r_{n_k} \to x \). For this note that if \( \varepsilon > 0 \) and \( N > 0 \), \( (x - \varepsilon, x + \varepsilon) \cap \{r_n : n \geq 1\} \) is an infinite set and so

(2) there is an \( n > N \) so that \( |r_{n_k} - x| < \varepsilon \).

Now proceed inductively to construct \( n_1 < n_2 < \ldots < n_k < \ldots \) so that for all \( k \) \( |r_{n_k} - x| < 1/k \). This is clearly possible for \( k = 1 \) by the above. Assume we have \( n_1 < n_2 < \ldots < n_k \) so that \( |r_{n_j} - x| < 1/j \) for \( j \leq k \). By (2) we may choose \( n_{k+1} > n_k \) so that \( |r_{n_{k+1}} - x| < 1/(k+1) \). This completes the inductive construction. Clearly \( r_{n_k} \to x \) and so \( x \in L(\{r_n\}) \). This shows that the latter set contains \([1,2]\) and so the argument is complete.

So by Thm. 5.4 we see that \( \limsup r_n = \max [1,2] = 2 \) and \( \liminf r_n = \min [1,2] = 1 \), which you can also derive directly from the definitions of \( \limsup \) and \( \liminf \).

**Example 2.** Let \( x_n = (-1)^n (1 + n^{-1}) \). Find \( \limsup x_n \) and \( \liminf x_n \).

Note that \( x_{2n} = 1 + (2n)^{-1} \) and \( x_{2n+1} = -(1 + (2n - 1)^{-1}) \). Since \( x_{2n} \) is decreasing in \( n \) and \( x_{2n-1} < 0 \), it follows easily that \( \bar{x}_{2n} = \sup \{x_k : k \geq 2n\} = 1 + (2n)^{-1} \to 1 \), and so \( \limsup x_n = \lim_{n \to \infty} \bar{x}_n = \lim_{n \to \infty} \bar{x}_{2n} = 1 \). (The fact that \( \bar{x}_n \) converges means the limit may be obtained along the subsequence \( \bar{x}_{2n} \).) Similar reasoning gives \( \underline{x}_{2n-1} = -(1 + (2n - 1)^{-1}) \), and so \( \liminf x_n = \lim_{n \to \infty} \underline{x}_n = \lim_{n \to \infty} \underline{x}_{2n-1} = -1 \).