Q7(a) For all \( i \leq n \), \( A_i \subseteq B_n \) and so \( \bar{A}_i \subseteq \bar{B}_n \). Taking the union over \( i \) gives \( \bigcup_{i=1}^{n} \bar{A}_i \subseteq \bar{B}_n \). For the reverse inclusion note that \( \bigcup_{i=1}^{n} A_i \) is closed because it is a finite union of closed sets. So since it contains \( B_n = \bigcup_{i=1}^{n} A_i \), it will contain \( \bar{B}_n \).

(b) This is proved exactly as in the first part of (a).

Let \( A_i = [0, 1 - i^{-1}] \). Then \( B = \bigcup_{i=1}^{\infty} A_i = [0, 1] \) so \( \bar{B} = [0, 1] \). On the other hand \( \bigcup_{i=1}^{\infty} \bar{A}_i = \bigcup_{i=1}^{\infty} [0, 1 - i^{-1}] = [0, 1) \), which is a proper subset of \( [0, 1] = \bar{B} \).

Q8. Yes. Let \( E \) be open and \( p \in E \). Choose \( r_0 > 0 \) so that \( N_{r_0}(p) \subseteq E \). Then for any \( 0 < r \leq r_0 \), \( N_r(p) \cap E = N_r(p) \) which is infinite and so contains points other than \( p \). If \( r \geq r_0 \), the previous set is larger and so still contains points other than \( p \). This proves \( p \) is a limit point of \( E \).

If \( E \) is closed this is not necessarily true. Take \( E = \{ x_0 \} \) in any metric space (such as \( \mathbb{R}^2 \)). Then \( E \) is closed as we proved in class. But for any \( r > 0 \), \( N_r(x_0) \cap E = \{ x_0 \} \) contains no points in \( E \) other than \( x_0 \) and so \( x_0 \) is not a limit point of \( E \).

Q9 (d) \( p \in (E^c)^c \) iff \( \forall r > 0 \, N_r(p) \not\subseteq E \) iff \( \forall r > 0 \, N_r(p) \cap E^c \neq \emptyset \) iff \( p \in \overline{E^c} \).

(e) No. Consider \( E = \mathbb{Q} \) as a subset of the metric space \( \mathbb{R} \). Then \( \overline{\mathbb{Q}} = \mathbb{R} \) (recall \( \mathbb{Q} \) is dense in \( \mathbb{R} \) and so \( \mathbb{Q} \) has interior in \( \mathbb{R} \)). But \( \mathbb{Q} \) has empty interior. To see this assume \( p \in \mathbb{Q}^c \). Then there is an \( r \) so that \( (p - r, p + r) \subseteq \mathbb{Q} \). But any non-empty open interval will contain an irrational (*) so this contradiction shows that \( \mathbb{Q} \) has empty interior.

(*)Proof. Let \( (a, b) \) be non-empty. Choose a rational \( r \in (a/\sqrt{2}, b/\sqrt{2}) \). Then \( r\sqrt{2} \) is an irrational number (why?) in \( (a, b) \).

(f) No. Take \( E = \mathbb{Q} \) again. By the above, \( \mathbb{Q}^c = \emptyset \) and so has empty closure, but \( \mathbb{Q} \) has closure equal to \( \mathbb{R} \).

Q11. \( d_1 \) is not a metric: \( d_1(0, 1) + d_1(1, 2) = 2 < 4 = d_1(0, 2) \), so the triangle \( \leq \) fails.

\( d_2 \) is a metric. Clearly \( d_2 \) is symmetric and \( d_2(x, y) = 0 \) iff \( x = y \). Finally,

\[
(\sqrt{d_2(x, z)} + \sqrt{d_2(y, z)})^2 \geq d_2(x, y)^2 + d_2(y, z)^2 = |x - y| + |y - z| \geq |x - z| = d_3(x, z)^2,
\]

and take square roots of both sides to prove the triangle inequality.

\( d_3 \) is not a metric. \( d_3(1, -1) = 0 \) but \( 1 \neq -1 \).

\( d_4 \) is not a metric. \( d_4(2, 1) = 0 \) but \( 1 \neq 2 \).

\( d_5 \) is a metric. Symmetry of \( d_5 \) is obvious as is \( d_5(x, y) = 0 \) iff \( x = y \). Note that \( f(x) = \frac{x}{\sqrt{1 + x}} \) is an increasing concave down \( (f' \text{ is decreasing}) \) function on \( [0, \infty) \). This shows that for \( x, y \geq 0 \),

\[
f(x + y) - f(y) = \int_{y}^{x+y} f'(t)dt \leq \int_{0}^{x} f'(t)dt = f(x). \text{ Therefore (1) } f(x + y) \leq f(x) + f(y) .
\]

So set \( a = |x - y|, b = |y - z| \) and \( c = |x - z| \). Then by (1) above

\[
d_5(x, y) + d_5(y, z) = f(a) + f(b) \geq f(a + b) \geq f(c) = d_5(x, z) .
\]

the next to last inequality follows from the fact that \( f \) is increasing and \( a + b \geq c \) by the ordinary triangle inequality.

You could also do a brute force check of the triangle inequality.