1. Let $X = \{(x,y) \in \mathbb{R}^2 : (x,y) \neq (0,0)\}$. Define $f : X \rightarrow \mathbb{R}$ by $f(x,y) = \frac{xy}{x^2+y^2}$. Prove that $f$ is continuous on $X$ but that it is not possible to define $f(0,0)$ to make $f$ continuous on $\mathbb{R}^2$.

Solution. Suppose we can define $f(0,0)$ so that $f$ becomes continuous at $(0,0)$. Let $z_n = (2^{-n},2^{-n}) \rightarrow (0,0)$ and $w_n = (2^{-n},0) \rightarrow (0,0)$. Then $f(z_n) = \frac{1}{2}$ and $f(w_n) = 0$. But $\lim_{n \rightarrow \infty} f(z_n) = f(0,0) = \lim_{n \rightarrow \infty} f(w_n)$, which implies $1/2 = 0$. Hence it is not possible to define $f(0,0)$ so that $f$ becomes continuous at $(0,0)$.

2. Prove that the function $f : [0,\infty) \rightarrow \mathbb{R}$ given by $f(x) = \sqrt{x}$ is uniformly continuous.

Solution. Let $x', x \geq 0$ and assume wlog that $x' \geq x$. Then $(\sqrt{x'} - \sqrt{x})^2 = x' + x - 2\sqrt{x'}\sqrt{x} \leq x' + x - 2\sqrt{x} = x' - x = |x' - x|$. Let $\varepsilon > 0$ and take $\delta = \varepsilon^2$. By the above we see that if $x, x' \geq 0$ satisfy $|x' - x| < \delta$, then $|\sqrt{x'} - \sqrt{x}| \leq \sqrt{|x' - x|} < \sqrt{\varepsilon^2} = \varepsilon$. This establishes the uniform continuity of the square root function on $[0,\infty)$.

3. Let $p : \mathbb{R} \rightarrow \mathbb{R}$ be an odd degree polynomial, that is $p(x) = \sum_{k=0}^{2N+1} a_k x^k$, where $a_{2N+1} \neq 0$ and $N \in \mathbb{N}$. Prove that $p$ has a real root, i.e. there is an $x_0 \in \mathbb{R}$ such that $p(x_0) = 0$.

Solution. By replacing $p$ with $-p$ we may assume that $a_{2N+1} > 0$. Using Theorem 4.34 in Rudin we have

$$\lim_{x \rightarrow +\infty} \frac{p(x)}{x^{2N+1}} = \sum_{k=0}^{2N+1} a_k \lim_{x \rightarrow +\infty} x^{k-2N-1} = a_{2N+1} > 0,$$

the last equality because all the limits are 0 except for the $k = 2N+1$ term where the limit is 1. Take $\varepsilon = a_{2N+1}/2 > 0$ to see that there is an $R_0 > 0$ s.t. if $x \geq R_0$, then $p(x)/x^{2N+1} > a_{2N+1}/2$, which implies that $p(x) > 0$. Similarly we have $\lim_{x \rightarrow -\infty} \frac{p(x)}{x^{2N+1}} = a_{2N+1} > 0$ and so there is a $R_1 < 0$ s.t. for $x \leq R_1 < 0$, $\frac{p(x)}{x^{2N+1}} > a_{2N+1}/2$ which implies that $p(x) < a_{2N+1}/2 x^{2N+1} < 0$. So as $p(R_1) < 0 < p(R_0)$, the Intermediate Value Theorem implies ($p$ is continuous since it is a polynomial) there is an $x_0 \in [R_1, R_0]$ s.t. $p(x_0) = 0$. 


4. Prove that a uniformly continuous function \( f : (0,1) \to \mathbb{C} \) is bounded.

Solution. By uniform continuity there is a natural number \( N \) such that if \( |x - x'| < 1/N \) (\( x, x' \) in \( (0,1) \)), then \( |f(x) - f(x')| < 1 \). For \( i = 1, 2, \ldots, N \), let \( x_i \) be the midpoint of \( ((i - 1)/N, i/N) \). Let \( M = \max(|f(x_1)|, \ldots, |f(x_N)|) + 1 \). Then for any \( x \in (0,1) \) we may choose \( i \) so that \( |x - x_i| < 1/N \) (e.g., choose \( i \) so that \( x \in ((i - 1)/N, i/N) \)). Then by the triangle inequality,

\[
|f(x)| \leq |f(x) - f(x_i)| + |f(x_i)| < 1 + |f(x_i)| \leq M.
\]

Therefore \( |f(x)| \) is bounded by \( M \) on \( (0,1) \).

5. Let \( X \) be a compact metric space, let \( Y \) be a metric space, and let \( f : X \to Y \) be continuous. Let \( \{F_n\} \) be a decreasing sequence of nonempty closed subsets of \( X \). Prove that \( \cap_n f(F_n) = f(\cap_n F_n) \).

Solution. Assume \( y \in \cap_n f(F_n) \). Then for any natural number \( n \) there is an \( x_n \in F_n \) such that \( y = f(x_n) \). Since \( \{x_n\} \) is a sequence in the compact space \( X \), there is a convergent subsequence \( x_{n_k} \to x \in X \). Fix \( n_0 \). Then \( \{x_{n_k} : n_k \geq n_0\} \) is a sequence taking values in the closed set \( F_{n_0} \) and so the limit of this sequence, \( x \), must belong to \( F_{n_0} \). As \( n_0 \) is arbitrary we have proved that \( x \in \cap_n F_n \). By continuity of \( f \) we have \( y = \lim_{k \to \infty} f(x_{n_k}) = f(x) \) and so \( y \in f(\cap_n F_n) \). This proves that \( \cap_n f(F_n) \subset f(\cap_n F_n) \). The converse inclusion is true for any sets \( F_n \), and so we in fact obtain \( f(\cap_n F_n) = \cap_n f(F_n) \).

6. Chapter 4, Questions #2, 12, 14, 18, 21

7. Let \( E \subset X \), where \( (X, d) \) is a metric space.

(a) If \( \bar{A}^E \) denotes the relative closure of a subset \( A \) of \( E \) in \( E \), and \( \bar{A} \) denotes the closure of \( A \) in \( X \), prove that \( \bar{A}^E = \bar{A} \cap E \). (You may the use the fact that for \( F \subset E \), \( F \) is relatively closed in \( E \) iff \( F = E \cap V \) for some closed set \( V \) in \( X \), as this follows easily from Theorem 2.30 in the text by taking complements in the appropriate space.)

Solution. Since \( \bar{A} \cap E \) is closed in \( E \), it follows that \( \bar{A}^E \subset \bar{A} \cap E \). Let \( F \) be a closed set in \( E \) such that \( A \subset F \). Then there is a closed subset \( V \) of \( X \) s.t. \( F = V \cap E \). Therefore \( A \subset V \) which implies that \( \bar{A} \subset V \), and hence \( \bar{A} \cap E \subset V \cap E = F \). Taking \( F = \bar{A}^E \) we get \( \bar{A} \cap E \subset \bar{A} \cap E \) and so equality holds.
We say that a metric space $X$ is connected iff it is a connected subset of itself. Prove that if $E \subset X$, then $E$ is a connected subset of $X$ iff $E$ is a connected metric space (when equipped with the metric it inherits from $X$). So as for compactness, connectedness is an absolute, not a relative, property.

Solution. We show that $E$ is a disconnected metric space iff $E$ is a disconnected subset of $X$.

$(\Rightarrow)$ We have $E = A \cup B$, where $A$, $B$ are non-empty and $A \cap \bar{B}^E = A^E \cap B = \emptyset$. From (a) we see that this implies $A \cap E \cap \bar{B} = \emptyset$. Since $A, B \subset E$, we conclude that $A \cap \bar{B} = \emptyset$ and so $E$ is a disconnected subset of $X$.

$(\Leftarrow)$ We have $E = A \cup B$, where $A$, $B$ are non-empty and $\bar{A} \cap \bar{B} = A \cap \bar{B} = \emptyset$. From (a) we see that $\bar{A} \cap B = A \cap E \cap B = \emptyset$ and symmetrically, $\bar{B} \cap A = A \cap \bar{B} = \emptyset$. This implies $E$ is a disconnected metric space.

(c) Prove the following are equivalent:

i. $X$ is a connected metric space.

ii. $X$ cannot be written as the union of two disjoint open non-empty sets.

iii. The only subsets of $X$ which are both open and closed are $\emptyset$ and $X$.

Solution. We prove the equivalence of the negations of i., ii., iii.

(not i. $\Rightarrow$ not ii.) We have $X = A \cup B$, where $A$, $B$ are non-empty and $A \cap \bar{B} = \emptyset$. This implies that $B \subset (\bar{A})^c$ and $A \subset (\bar{B})^c$ and hence that $(\bar{A})^c$ and $(\bar{B})^c$ are non-empty. Note also that $(\bar{A})^c \cap (\bar{B})^c = (\bar{A} \cup \bar{B})^c = X^c = \emptyset$. Therefore $X = (\bar{A})^c \cup (\bar{B})^c$ shows that $X$ is the union of two non-empty disjoint open subsets.

(not ii. $\Rightarrow$ not iii.) Assume $X = G_1 \cup G_2$ where $G_i$ is non-empty open, and $G_1 \cap G_2$ is empty. Then $G_1 = G_2^c$ is also closed (as $G_2$ is open). Since $G_1^c = G_2$ is non-empty we see that $G_1$ is distinct from $X$ and $\emptyset$, and is both open and closed.

(not iii. $\Rightarrow$ not i.) Let $A$ be a set which is open and closed and is distinct from $X$ and $\emptyset$. Then $A^c$ is also closed and open and non-empty. Therefore $X = A \cup A^c$, $A$ and $A^c$ are non-empty, $\bar{A} \cap A^c = A \cap A^c = \emptyset$, and $A \cap \bar{A}^c = A \cap A^c = \emptyset$. This shows that $X$ is a disconnected metric space.