MATHEMATICS 320, FALL 2016, PROBLEM SET 6
Due on Monday, November 28, in class

Write clearly and legibly, in complete sentences. You must provide complete explanations for all your solutions; answers without justification, even if correct, will not be marked. You may discuss the homework with other students, but the final write-up must be your own.

1. Find all complex numbers $z$ such that the following power series converge:
   
   (a) $\sum_{n=1}^{\infty} 3^{n+\frac{1}{2}} n^{-2} z^n$.
   
   (b) $\sum_{n=0}^{\infty} 2^{-n} z^{2^n}$.

2. Let $a_n \geq 0$ for all $n \in \mathbb{N}$. Prove that
   
   $$\sum_{n=1}^{\infty} a_n = \sup \left\{ \sum_{n \in F} a_n : F \subset \mathbb{N} \text{ finite} \right\}.$$ 

   Is this still true without the assumption that $a_n \geq 0$? Prove your answer.

3. Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series such that $a_n > 0$, $b_n > 0$, and
   
   $$\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}$$

   for all $n \in \mathbb{N}$. Prove that if $\sum_{n=1}^{\infty} b_n$ converges, then also $\sum_{n=1}^{\infty} a_n$ converges.

4. Decide whether the following series are convergent or divergent. Prove your answers.
   
   (a) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$
   
   (b) $\sum_{n=1}^{\infty} \left( \frac{n}{n+1} \right)^n$

5. (a) Let $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$. Prove that the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series $\sum_{k=1}^{\infty} k a_k^2$ converges.
(b) Apply (a) to determine if the series \( \sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}} \) converges or diverges.

6. The following questions from the textbook should be done but are NOT to be handed in: Chapter 3 # 6, 7, 8.