1. (14 marks) If $E$ is a subset of a metric space $X$, define the boundary of $E$, $\partial E$, by
$$\partial E = \{ x \in X : \forall r > 0, N_r(x) \cap E \neq \emptyset \text{ and } N_r(x) \cap E^c \neq \emptyset \}.$$ 

(a) (3 marks) Prove that $\partial E = \overline{E} - E^\circ$.

$$x \in \partial E \iff \forall r > 0 \ N_r(x) \cap E \neq \emptyset, \text{ and } \forall r > 0 \ N_r(x) \cap E^c \neq \emptyset$$
$$\iff x \in \overline{E}, \text{ and } \forall r > 0 \ N_r(x) \not\subseteq E$$
$$\iff x \in \overline{E}, \text{ and } x \not\in E^\circ$$
$$\iff x \in \overline{E} - E^\circ.$$ 

(b) (2 marks) Prove that $E$ is open iff $E \cap \partial E = \emptyset$.

Note by (a), that $E \cap \partial E = E \cap (\overline{E} - E^\circ) = E \cap \overline{E} \cap (E^\circ)^c = E \cap (E^\circ)^c$. Therefore $E \cap \partial E = \emptyset$ iff $E \subseteq E^\circ$, that is, iff $E$ is open.

(c) (2 marks) Prove that $E$ is closed iff $\partial E \subseteq E$.

Note by its definition we have $\partial E = \partial(E^c)$. So using this and (b) we have: $E$ is closed iff $E^c$ is open iff $E^c \cap \partial(E^c) = \emptyset$ iff $E^c \cap \partial E = \emptyset$ iff $\partial E \subseteq E$.

(d) (2 marks) If $X = \mathbb{R}$, find $\partial \mathbb{Q}$.

Claim that $\partial \mathbb{Q} = \mathbb{R}$. Both the rationals and irrationals are dense in the real numbers. Therefore for any real number $x$, $N_r(x) \cap \mathbb{Q} \neq \emptyset$ and $N_r(x) \cap (\mathbb{Q}^c) \neq \emptyset$. This implies that $x \in \partial \mathbb{Q}$ and so proves the claim.

(e) (5 marks) If $X = \mathbb{R}$, find $\partial [0, 1)$. If $X = \mathbb{C}$, find $\partial [0, 1)$.

First consider $X = \mathbb{R}$. Then $[0, 1] = [0, 1]$ and $[0, 1)^{\circ} = (0, 1)$. By (a), this implies that $\partial [0, 1) = [0, 1] - (0, 1) = \{0, 1\}$.

Next consider $X = \mathbb{C}$ which as a metric space is just $\mathbb{R}^2$. In this case we claim that $\partial [0, 1) = [0, 1]$.

Recall our convention for identifying the real number $x$ with the
complex number \((x, 0)\), which means, for example, that
\([0, 1) = \{(x, 0) : 0 \leq x < 1\}\). If \((x, 0) \in [0, 1]\), then
\(N_r((x, 0)) \cap [0, 1) = ((x - r, x + r) \cap \{x' : 0 \leq x' < 1\}) \times \{0\} \neq \emptyset\),
where in the last line we use the fact that \(0 \leq x < 1\) and an
elementary argument. We also have for \(x\) as above, \(N_r((x, 0)) \cap [0, 1)^c\) contains \((x, r/2)\) and so is also non-empty. This shows that
\([0, 1) \subset \partial (0, 1]\).

On the other hand if \((x, y) \in \partial [0, 1) \subset [0, 1)\) (the last by (a)), then
there is a sequence \(0 \leq x_n < 1\) so that \((x_n, 0) \to (x, y)\). The latter
implies that \(y = 0\) and \(x_n \to x\) (as real numbers) and so implies
\(0 \leq x \leq 1\). This proves that \((x, y) = (x, 0) \in [0, 1]\) and so shows
that \(\partial [0, 1) \subset [0, 1]\), completing the proof.

2. (10 marks)

Let \(c_0\) be the space of real-valued sequences \(\{x_n\}\) which converge to zero,
equipped with the metric \(d(\{x_n\}, \{y_n\}) = \sup_n |x_n - y_n|\). The fact that
\(d\) is a metric on \(c_0\) follows from Q 3(a) on Problem Set 4.

(a) (7 marks) Let \(e_k\) denote the sequence in \(c_0\) which is identically
0, except for the \(k\)th entry which equals 1. Prove that \(\{e_k\}\) is
a bounded sequence in \(c_0\) (i.e., it takes values in a bounded set)
which has no convergent subsequence.

Let \(e_k(n)\) denote the \(n\)th entry of the sequence \(e_k\) for each natural
number \(n\). Note that if \(k \neq \ell\) then \(d(e_k, e_\ell) = \sup\{|e_k(n) - e_\ell(n)| : n \in \mathbb{N}\} = 1\)
because the difference inside the supremum equals 1 if
\(n = k\) or \(\ell\) and is 0 otherwise. Now let \(\{e_n\}\) be any subsequence of
\(\{e_k\}\). If \(j \neq k\), then \(n_j \neq n_k\) and so by the above \(d(e_{n_j}, e_{n_k}) = 1\).
This shows that \(\{e_{n_k} : k \in \mathbb{N}\}\) is not a Cauchy sequence in \(c_0\) and
so cannot converge in \(c_0\). Finally \(d(e_n, 0) = 1\) where 0 is as in
(b). This shows that \(\{e_n : n \in \mathbb{N}\} \subset N_2(0)\) (it also shows it takes
values in the closed unit ball \(B\) in (b)) and so \(\{e_n\}\) is a bounded
sequence with no convergent subsequence.

(b) (3 marks) Prove that the closed unit ball in \(c_0\), \(B = \{p \in c_0 : d(0, p) \leq 1\}\) (here 0 denotes the sequence consisting of all 0’s) is
not compact.

As noted in the proof of (a), \(\{e_n\}\) is a sequence taking values in
\(B\). As we have just proved it has no convergent subsequence, \(B\)
3. (10 marks) Prove that the metric space \((c_0, d)\) defined in the previous question is complete.

Let \(\{x_k\}\) be a Cauchy sequence in \(c_0\). As \(\{x_k\}\) is a sequence of sequences, it is important to get our notation right:

let \(x_k = \{x_k(n) : n \in \mathbb{N}\}\) (or one could use double subsequences and write \(x_k = \{x_{kn} : n \in \mathbb{N}\}\)). The Cauchy condition means

\[\forall \varepsilon > 0 \exists N(\varepsilon) \in \mathbb{N} \text{ s.t. } \ell, k \geq N(\varepsilon) \Rightarrow \sup\{|x_k(n) - x_\ell(n)| : n \in \mathbb{N}\} < \varepsilon.\]

(1)

This implies that for \(n\) fixed if \(\ell, k \geq N(\varepsilon)\), then \(|x_k(n) - x_\ell(n)| < \varepsilon\), and so \(\{x_k(n) : k \in \mathbb{N}\}\) is a Cauchy sequence in \(\mathbb{R}\). The completeness of \(\mathbb{R}\) shows there is a real number \(x(n)\) such that \(\lim_{k \to \infty} x_k(n) = x(n)\). For a fixed \(k \geq N(\varepsilon)\) and \(n \in \mathbb{N}\), by (1) we have for \(\ell \geq N(\varepsilon)\),

\[|x_k(n) - x_\ell(n)| < \varepsilon.\]

Now let \(\ell \to \infty\) in this inequality to deduce that

\[|x_k(n) - x(n)| \leq \varepsilon \text{ for all } n, \text{ and therefore}\]

\[\forall \varepsilon > 0 \forall k \geq N(\varepsilon), \sup\{|x_k(n) - x(n)| : n \in \mathbb{N}\} \leq \varepsilon.\]

(2)

We next show that \(x \in c_0\). Fix \(\varepsilon > 0\). Then \(x_{N(\varepsilon)} \in c_0\), so there is an \(N_0\) s.t. \(n \geq N_0\) implies \(|x_{N(\varepsilon)}(n)| < \varepsilon\). Combine this with (2) with \(k = N(\varepsilon)\) to conclude that for \(n \geq N_0\),

\[|x(n)| \leq |x(n) - x_{N(\varepsilon)}(n)| + |x_{N(\varepsilon)}(n)| < 2\varepsilon.\]

This proves that \(x \in c_0\). Finally (2) shows that for any \(\varepsilon > 0\) and all \(k \geq N(\varepsilon)\), \(d(x_k, x) \leq \varepsilon\). This shows that \(x_k \to x\) in \(c_0\). Therefore \(c_0\) is complete.

4. (8 marks) Evaluate the following and justify your answers:

(a) \(\limsup_{n \to \infty} (-1)^n \frac{n^2 + 1}{2n^2 + 1}\).

**Lemma.** If \(a_n \geq 0\) and \(a_n \to a\), then \(\limsup_{n \to \infty} (-1)^n a_n = a\) and \(\liminf_{n \to \infty} a_n = -a\).

**Proof.** It suffices to show that the set of subsequential limits of \(\{(-1)^n a_n\}\), \(\mathcal{L}\), is \((-a, a)\) (of course \(a\) may be zero). Clearly \((-1)^{2n} a_{2n} = a_{2n} \to a\) and \((-1)^{2n-1} a_{2n-1} = -a_{2n-1} \to -a\) so
\{-a, a\} \subset L. Assume \((-1)^{n_k}a_{n_k} \to \ell\). Then \(a_{n_k} = |(-1)^{n_k}a_{n_k}| \to |\ell|\) (e.g. by the triangle inequality). But as the left-hand side also converges to \(a\) we have \(a = |\ell|\) and so \(\ell = \pm a\). This proves \(L \subset \{-a, a\}\) and the proof is complete. \(\square\)

We have \(a_n = \frac{n^2 + 1}{2n^2 + 1} = \frac{1}{2} + \frac{1}{4n^2 + 2} \to 1\), and so by the Lemma,
\[
\limsup_{n \to \infty} (−1)^n \frac{n^2 + 1}{2n^2 + 1} = 1 \text{ (and } \liminf_{n \to \infty} (−1)^n \frac{n^2 + 1}{2n^2 + 1} = −1 \text{ although this wasn’t asked}).
\]

(b) \(\liminf_{n \to \infty} \frac{\sin(\pi n/8)n^n}{n!}\).

We first show that \(\frac{n^n}{n!} \to \infty\). Note that \(\frac{n^n}{n!} = \frac{n}{n} \times \frac{n}{n-1} \times \cdots \times \frac{n}{1} \geq \frac{n}{1} = n\). As \(n \to +\infty\) this implies that \(\frac{n^n}{n!} \to \infty\).

Note that if \(n_k = 12 + 16k\), then \(\frac{\sin(\pi n_k/8)n_k^n}{n_k!} = -\frac{n_k^n}{n_k!} \to -\infty\) by the above. The fact that \(-\infty\) is in the set of subsequential limits, \(L\), of \(\{\frac{\sin(\pi n/8)n^n}{n!}\}\) implies that \(\liminf_{n \to \infty} \frac{\sin(\pi n/8)n^n}{n!} = \min L = -\infty\).

5. (8 marks) If \(\{a_n\}\) and \(\{b_n\}\) are real-valued sequences, and \(\{b_n\}\) is bounded, prove that
\[
\limsup_{n \to \infty} (b_n - a_n) \leq \limsup_{n \to \infty} b_n - \liminf_{n \to \infty} a_n.
\]

Choose a subsequence so that \(b_{n_k} - a_{n_k} \to \limsup_{n \to \infty} (b_n - a_n) \equiv L\). Assume wlog that \(L > -\infty\) (or the result is immediate). As \(b_{n_k}\) is bounded it has a further subsequence \(b_{n_{kj}}\) converging to a real number \(b\). Note that \(b_{n_{kj}} - a_{n_{kj}} \to L\), so to control the number of subsequences we abuse notation and just write \(n_j\) for \(n_{kj}\) (usually we just say “by taking a further subsequence we may assume \(b_{n_k} \to b\)”).

Assume first \(L = \infty\). Then claim that \(a_{n_j} \to -\infty\). Let \(M\) be any real number and choose \(K\) so that \(|b_n| \leq K\) for all \(n\). Recalling that \(b_{n_j} - a_{n_j} \to \infty\), we may choose \(N\) so that \(j > N\) implies that \(b_{n_j} - a_{n_j} > -M + K\) which implies that \(a_{n_j} = b_{n_j} - (b_{n_j} - a_{n_j}) < K + M - K = M\), and so proves the claim. Now therefore \(\limsup_{n \to \infty} b_n - \liminf_{n \to \infty} a_n \geq -K - \lim_{j \to \infty} a_{n_j} = -K - (\infty) = \infty\). The result is immediate.

Assume finally that \(L\) is finite. Then \(a_{n_j} = b_{n_j} - (b_{n_j} - a_{n_j}) \to b - L\) and so \(L = \lim_{j \to \infty} b_{n_j} - \lim_{j \to \infty} a_{n_j} \leq \lim \sup b_n - \lim \inf a_n\).