1. If $E$ is a subset of a metric space $X$, define the boundary of $E$, $\partial E$, by
   \[ \partial E = \{ x \in X : \forall r > 0, N_r(x) \cap E \neq \emptyset \text{ and } N_r(x) \cap E^c \neq \emptyset \} . \]
   (a) Prove that $\partial E = \overline{E} - E^o$.
   (b) Prove that $E$ is open iff $E \cap \partial E = \emptyset$.
   (c) Prove that $E$ is closed iff $\partial E \subset E$.
   (d) If $X = \mathbb{R}$, find $\partial \mathbb{Q}$.
   (e) If $X = \mathbb{R}$, find $\partial [0, 1)$. If $X = \mathbb{C}$, find $\partial [0, 1)$.

2. Let $c_0$ be the space of real-valued sequences $\{x_n\}$ which converge to zero, equipped with the metric $d(\{x_n\}, \{y_n\}) = \sup_n |x_n - y_n|$. The fact that $d$ is a metric on $c_0$ follows from Q 3(a) on Problem Set 4.
   (a) Let $e_k$ denote the sequence in $c_0$ which is identically 0, except for the $k$th entry which equals 1. Prove that $\{e_k\}$ is a bounded sequence in $c_0$ (i.e., it takes values in a bounded set) which has no convergent subsequence.
   (b) Prove that the closed unit ball in $c_0$, $B = \{ p \in c_0 : d(0, p) \leq 1 \}$ (here 0 denotes the sequence consisting of all 0’s) is not compact.

3. Prove that the metric space $(c_0, d)$ defined in the previous question is complete.

4. Evaluate the following and justify your answers:
   (a) $\limsup_{n \to \infty} (-1)^n \frac{n^2 + 1}{2n^2 + 1}$.
   (b) $\liminf_{n \to \infty} \frac{\sin(\pi n / 8)n^n}{n!}$.

5. If $\{a_n\}$ and $\{b_n\}$ are real-valued sequences, and $\{b_n\}$ is bounded, prove that
   \[ \limsup_{n \to \infty} (b_n - a_n) \leq \limsup_{n \to \infty} b_n - \liminf_{n \to \infty} a_n . \]

6. The following questions from the textbook should be done but are NOT to be handed in: Chapter 2 # 23, 24, 25, 26; Chapter 3 # 4.