1. (6 marks) For any two non-empty sets $E, F \subset \mathbb{R}$, define $\text{dist}(E, F) = \inf\{|x - y| : x \in E, y \in F\}$. Let $E, F \subset \mathbb{R}$ be two non-empty closed sets with $E$ bounded. Prove that there are points $x \in E$ and $y \in F$ such that $\text{dist}(E, F) = |x - y|$.

Let $d = \text{dist}(E, F)$. For each $n \in \mathbb{N}$, we can choose $x_n \in E$ and $y_n \in F$ such that $|x_n - y_n| \leq d + \frac{1}{n}$. Since $E \subset \mathbb{R}$ is bounded, the sequence $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$, with $x_{n_k} \to x$ for some $x \in \mathbb{R}$. Since $E$ is also closed, we have $x \in E$.

Let $z_k = x_{n_k}$ and $w_k = y_{n_k}$, to simplify notation. Then $z_k \in E$, $w_k \in F$, and $|z_k - w_k| \leq d + \frac{1}{n_k} \leq d + \frac{1}{k}$. Also, since $z_k \to x$, the sequence $\{z_k\}$ is bounded: $|z_k| \leq M$ for some $M > 0$. Then by the triangle inequality, $|w_k| \leq |z_k - w_k| + |w_k| \leq M + d + 1$ for all $k$.

The set $F \cap [-M - d - 1, M + d + 1] \subset \mathbb{R}$ is bounded and closed. By the same argument as above, the sequence $\{w_k\}$ has a convergent subsequence $\{w_{k_\ell}\}_{\ell=1}^{\infty}$, with $w_{k_\ell} \to y$ for some $y \in F$. Since $z_k \to x$, we also have $z_{k_\ell} \to x$.

By the triangle inequality, we have $|x - y| \leq |z_{k_\ell} - w_{k_\ell}| + |x - z_{k_\ell}| + |w_{k_\ell} - y| \leq d + \frac{1}{k_\ell} + |x - z_{k_\ell}| + |w_{k_\ell} - y|$. Since $k_\ell \to \infty$, $w_{k_\ell} \to y$ and $z_{k_\ell} \to x$, for any $\epsilon > 0$ we can choose $\ell$ large enough so that the last three terms are all less than $\epsilon/3$. Hence $|x - y| \leq d + \epsilon$. Since $\epsilon > 0$ was arbitrary, we have $|x - y| \leq d$. On the other hand, we also have $|x - y| \geq d$ from the definition of $d$. Therefore $|x - y| = d$.

2. (3 marks) In Chapter 2 #7 (below and not to hand in) you will prove (by giving a counterexample) that the formula $\bigcup_{j=1}^{\infty} E_j = \bigcup_{j=1}^{\infty} E_j$ is false— in fact you can do this with $X = \mathbb{R}$. Give an example of a metric space with infinitely many elements in which this formula is true for any sequence of sets $\{E_j\}$.

Let $X = \mathbb{Z}$, with the usual metric $d(x, y) = |x - y|$. We claim that for all $E \subset X$, $E = \overline{E}$. Indeed, let $E \subset X$ and suppose that $x$ is a limit point of $E$. Then there is an $y \in E$ such that $y \neq x$ and $|y - x| \leq 1/2$. But this is not possible, since the distance between any two distinct points in $X$ is at least 1.

Thus every subset of $X$ is closed. In particular, $\bigcup_{j=1}^{\infty} \overline{E_j} = \bigcup_{j=1}^{\infty} E_j = \bigcup_{j=1}^{\infty} E_j$.

3. Let $X$ be the set of all bounded sequences $\{x_n\}$ with real entries. We define a metric $d$ on $X$ by saying that $d(\{x_n\}, \{y_n\}) = \sup\{|x_n - y_n| : n \in \mathbb{N}\}$.

(a) (6 marks) Prove that this is a metric.

We first note that $d$ is well defined: if $\{x_n\}, \{y_n\}$ are bounded sequences with $|x_n| \leq K$ and $|y_n| \leq M$ for all $n \in \mathbb{N}$, then $|x_n - y_n| \leq K + M$, so the set $\{|x_n - y_n| : n \in \mathbb{N}\}$ is bounded and has a supremum.

- We have $|x_n - y_n| \geq 0$ always, hence $d(\{x_n\}, \{y_n\}) \geq 0$. If $d(\{x_n\}, \{y_n\}) = 0$, then $|x_n - y_n| = 0$ for all $n$, so that $\{x_n\} = \{y_n\}$. Also $d(\{x_n\}, \{x_n\}) = \sup 0 = 0$. 

4. (4 marks) Prove that $\mathbb{Q} \cap [0, 2]$ is not a compact subset of $\mathbb{R}$ by finding an open cover with no finite subcover.

Recall that $1 < \sqrt{2} < 2$ and $\sqrt{2}$ is irrational. Therefore the numbers $\sqrt{2} - 1, \sqrt{2} - \frac{1}{2}, \sqrt{2} - \frac{1}{3}, \ldots$ are all irrational and belong to $[0, \sqrt{2})$. Let

$$G_0 = (\sqrt{2}, 3), \quad G_1 = (-1, \sqrt{2} - 1), \quad G_n = \left(\sqrt{2} - \frac{1}{n - 1}, \sqrt{2} - \frac{1}{n}\right) \quad \text{for } n \in \mathbb{N}, \ n \geq 2.$$ 

Then $G_n$ are open, $\mathbb{Q} \cap [0, 2] \subset \bigcup_{n=0}^{\infty} G_n$, and this cover has no finite subcover (or, for that matter, any proper subcover).

Alternatively one could use the open cover of $\mathbb{Q} \cap [0, 2], \{V_n, n \in \mathbb{N}\}$, where $V_n = (-1, \sqrt{2} - n^{-1}) \cup (\sqrt{2} + n^{-1}, 3)$.

(b) (6 marks) Let $E_N$ be the set of all sequences $\{x_n\} \in X$ such that $x_n = 0$ for all $n \geq N$. Let $\{a_n\} \in X$. Prove that $\{a_n\}$ belongs to the closure of $\bigcup_{N \in \mathbb{N}} E_N$ if and only if $\lim_{n \to \infty} a_n = 0$.

Suppose that $\lim_{n \to \infty} a_n = 0$, and let $\epsilon > 0$. Then there is an $N \in \mathbb{N}$ such that for $n \geq N$ we have $|a_n| < \epsilon$. Let $\{b_n\}$ be the sequence defined by $b_n = a_n$ for $n < N$ and $b_n = 0$ for $n \geq N$. Then $\{b_n\} \in E_N \subset \bigcup_{N \in \mathbb{N}} E_N$, and $d(\{a_n\}, \{b_n\}) < \epsilon$. Since $\epsilon > 0$ was arbitrary, $\{a_n\}$ belongs to the closure of $\bigcup_{N \in \mathbb{N}} E_N$ as required.

Conversely, suppose that $\{a_n\}$ belongs to the closure of $\bigcup_{N \in \mathbb{N}} E_N$. We have to prove that $a_n \to 0$. Let $\epsilon > 0$, then there is a sequence $\{b_n\}$ such that $\{b_n\} \in E_N$ for some $N$ and $d(\{a_n\}, \{b_n\}) < \epsilon$. The last condition implies that $|a_n - b_n| < \epsilon$ for all $n \in \mathbb{N}$. But since $\{b_n\} \in E_N$, we have $b_n = 0$ for $n \geq N$, so that $|a_n| < \epsilon$ for all $n \geq N$. This implies that $a_n \to 0$. 

• $d(\{x_n\}, \{y_n\}) = \sup\{|x_n - y_n| : n \in \mathbb{N}\} = \sup\{|y_n - x_n| : n \in \mathbb{N}\} = d(\{y_n\}, \{x_n\})$ 

• Let $\{x_n\}, \{y_n\}, \{z_n\}$ be three sequences in $X$. Then for each $n \in \mathbb{N}$ we have $|x_n - z_n| \leq |x_n - y_n| + |y_n - z_n| \leq d(\{x_n\}, \{y_n\}) + d(\{y_n\}, \{z_n\})$. Taking the supremum in $n$, we get that $d(\{x_n\}, \{z_n\}) \leq d(\{x_n\}, \{y_n\}) + d(\{y_n\}, \{z_n\})$ as required.