1. A hemisphere $H$ and a portion $P$ of a paraboloid are shown. Suppose $F$ is a vector field on $\mathbb{R}^3$ whose components have continuous partial derivatives. Explain why

$$\int_H \text{curl} \mathbf{F} \cdot dS = \int_C \text{curl} \mathbf{F} \cdot d\mathbf{S}$$

2-6 Use Stokes' Theorem to evaluate $\int_C \text{curl} \mathbf{F} \cdot d\mathbf{S}$.

2. $F(x, y, z) = 2y \cos z \mathbf{i} + e^x \sin z \mathbf{j} + xe^z \mathbf{k}$, $S$ is the hemisphere $x^2 + y^2 + z^2 = 9$, $z \geq 0$, oriented upward.

3. $F(x, y, z) = x^2y^2 \mathbf{i} + y^2z^2 \mathbf{j} + xyz \mathbf{k}$, $S$ is the part of the paraboloid $z = x^2 + y^2$ that lies inside the cylinder $x^2 + y^2 = 4$, oriented upward.

4. $F(x, y, z) = x^2y \mathbf{i} + \sin(xyz) \mathbf{j} + 3y^2 \mathbf{k}$, $S$ is the part of the plane $x = 1$ that lies between the planes $y = 0$ and $y = 3$, oriented in the direction of the positive $y$-axis.

5. $F(x, y, z) = xz \mathbf{j} + x + y \mathbf{k}$, $S$ is the part of the cylinder $x^2 + y^2 = 1$ that lies inside the cone $z = x^2 + y^2$, oriented upward.

6. $F(x, y, z) = e^{-x} \cos z \mathbf{i} + x^2z \mathbf{j} + xyk$, $S$ is the hemisphere $x = \sqrt{1 - y^2 - z^2}$, oriented in the direction of the positive $x$-axis. [Hint: Use Equation 3.1]

7-10 Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$. In each case $C$ is oriented counterclockwise as viewed from above.

7. $F(x, y, z) = (x + y^2) \mathbf{i} + (y + z^2) \mathbf{j} + (z + x^2) \mathbf{k}$, $C$ is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.

8. $F(x, y, z) = e^{-x} \mathbf{i} + e^x \mathbf{j} + e^y \mathbf{k}$, $C$ is the boundary of the part of the plane $2x + y + 2z = 2$ in the first octant.

9. $F(x, y, z) = xz \mathbf{i} + 2xz \mathbf{j} + e^{y^2} \mathbf{k}$, $C$ is the circle $x^2 + y^2 = 16$, $z = 5$.

10. $F(x, y, z) = xy \mathbf{i} + 2z \mathbf{j} + 3y \mathbf{k}$, $C$ is the curve of intersection of the plane $x + z = 5$ and the cylinder $x^2 + y^2 = 9$.

11. (a) Use Stokes' Theorem to evaluate $\int_C F \cdot d\mathbf{r}$, where $F(x, y, z) = x^2z \mathbf{i} + xy^2 \mathbf{j} + z^2k$ and $C$ is the curve of intersection of the plane $x + y + z = 1$ and the cylinder $x^2 + y^2 = 9$ oriented counterclockwise as viewed from above.

(b) Graph both the plane and the cylinder with domains chosen so that you can see the curve $C$ and the surface that you used in part (a).

(c) Find parametric equations for $C$ and use them to graph $C$.

12. (a) Use Stokes' Theorem to evaluate $\int_C F \cdot d\mathbf{r}$, where $F(x, y, z) = x^2y \mathbf{i} + x^2 \mathbf{j} + 3x \mathbf{k}$ and $C$ is the curve of intersection of the hyperbolic paraboloid $z = y^2 - x^2$ and the cylinder $x^2 + y^2 = 1$ oriented counterclockwise as viewed from above.

(b) Graph both the hyperbolic paraboloid and the cylinder with domains chosen so that you can see the curve $C$ and the surface that you used in part (a).

(c) Find parametric equations for $C$ and use them to graph $C$.

13-15 Verify that Stokes' Theorem is true for the given vector field $\mathbf{F}$ and surface $S$.

13. $F(x, y, z) = y^2 \mathbf{i} + x \mathbf{j} + z^2 \mathbf{k}$, $S$ is the part of the paraboloid $z = x^2 + y^2$ that lies below the plane $z = 1$, oriented upward.

14. $F(x, y, z) = x \mathbf{i} + y \mathbf{j} + xyz \mathbf{k}$, $S$ is the part of the plane $2x + y + z = 2$ that lies in the first octant, oriented upward.

15. $F(x, y, z) = y \mathbf{i} + z \mathbf{j} + x \mathbf{k}$, $S$ is the hemisphere $x^2 + y^2 + z^2 = 1$, $y \geq 0$, oriented in the direction of the positive $y$-axis.

16. Let $C$ be a simple closed smooth curve that lies in the plane $x + y + z = 1$. Show that the line integral

$$\int_C zdx - 2y dy + 3y dz$$

depends only on the area of the region enclosed by $C$ and not on the shape of $C$ or its location in the plane.

17. A particle moves along line segments from the origin to the points $(1, 0, 0)$, $(1, 2, 1)$, $(0, 2, 1)$, and back to the origin under the influence of the force field

$$F(x, y, z) = z^2 \mathbf{i} + 2xy \mathbf{j} + 4y^2 \mathbf{k}$$

Find the work done.
18. Evaluate
\[ \int_C (y + \sin x) \, dx + (x^2 + \cos y) \, dy + x^3 \, dz \]
where \( C \) is the curve \( r(t) = (\sin t, \cos t, \sin 2t), \ 0 \leq t \leq 2\pi \).

**Hint:** Observe that \( C \) lies on the surface \( z = 2xy \).

19. If \( S \) is a sphere and \( F \) satisfies the hypotheses of Stokes' Theorem, show that \( \iint_S \text{curl} \, F \cdot dS = 0 \).

20. Suppose \( S \) and \( C \) satisfy the hypotheses of Stokes' Theorem and \( f, g \) have continuous second-order partial derivatives. Use Exercises 24 and 26 in Section 16.5 to show the following.

(a) \( \int_C (f \, \nabla g) \cdot dr = \iint_S (\nabla f \times \nabla g) \cdot dS \)

(b) \( \iint_C (f \, \nabla f) \cdot dr = 0 \)

(c) \( \iint_S (f \, \nabla g + g \, \nabla f) \cdot dS = 0 \)

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**Writing Project**

Although two of the most important theorems in vector calculus are named after George Green and George Stokes, a third man, William Thomson (also known as Lord Kelvin), played a large role in the formulation, dissemination, and application of both of these results. All three men were interested in how the two theorems could help to explain and predict physical phenomena in electricity and magnetism and fluid flow. The basic facts of the story are given in the margin notes on pages 1092 and 1129.

Write a report on the historical origins of Green's Theorem and Stokes' Theorem. Explain the similarities and relationship between the theorems. Discuss the roles that Green, Thomson, and Stokes played in discovering these theorems and making them widely known. Show how both theorems arose from the investigation of electricity and magnetism and were later used to study a variety of physical problems.


the origin. [This is a special case of Gauss’s Law (Equation 17.7.11) for a single charge. The relationship between \( \varepsilon \) and \( \varepsilon_0 \) is \( \varepsilon = 1/(4\pi\varepsilon_0) \).]

Another application of the Divergence Theorem occurs in fluid flow. Let \( \mathbf{v}(x, y, z) \) be the velocity field of a fluid with constant density \( \rho \). Then \( \mathbf{F} = \rho \mathbf{v} \) is the rate of flow per unit area. If \( P_0(x_0, y_0, z_0) \) is a point in the fluid and \( B_r \) is a ball with center \( P_0 \) and very small radius \( r \), then \( \text{div} \mathbf{F}(P) \approx \text{div} \mathbf{F}(P_0) \) for all points in \( B_r \) since \( \text{div} \mathbf{F} \) is continuous. We approximate the flux over the boundary sphere \( S \), as follows:

\[
\iiint_{S_0} \mathbf{F} \cdot d\mathbf{S} = \iiint_{B_r} \text{div} \mathbf{F} \, d\mathbf{V} = \iiint_{B_r} \text{div} \mathbf{F}(P_0) \, d\mathbf{V} = \text{div} \mathbf{F}(P_0) \iiint_{S_0} d\mathbf{S}
\]

This approximation becomes better as \( r \to 0 \) and suggests that

\[
\text{div} \mathbf{F}(P_0) = \lim_{r \to 0} \frac{1}{V(B_r)} \iiint_{S_0} \mathbf{F} \cdot d\mathbf{S}
\]

Equation 8 says that \( \text{div} \mathbf{F}(P_0) \) is the net rate of outward flux per unit volume at \( P_0 \). (This is the reason for the name divergence.) If \( \text{div} \mathbf{F}(P) > 0 \), the net flow is outward near \( P \) and \( P \) is called a source. If \( \text{div} \mathbf{F}(P) < 0 \), the net flow is inward near \( P \) and \( P \) is called a sink.

For the vector field in Figure 4, it appears that the vectors that end near \( P_1 \) are shorter than the vectors that start near \( P_1 \). Thus the net flow is outward near \( P_1 \), so \( \text{div} \mathbf{F}(P_1) > 0 \) and \( P_1 \) is a source. Near \( P_2 \), on the other hand, the incoming arrows are longer than the outgoing arrows. Here the net flow is inward, so \( \text{div} \mathbf{F}(P_2) < 0 \) and \( P_2 \) is a sink. We can use the formula for \( \mathbf{F} \) to confirm this impression. Since \( \mathbf{F} = x^2 \mathbf{i} + y^2 \mathbf{j} \), we have \( \text{div} \mathbf{F} = 2x^2 + 2y^2 \), which is positive when \( y > -x \). So the points above the line \( y = -x \) are sources and those below are sinks.

![Figure 4](image-url)

**FIGURE 4**
The vector field \( \mathbf{F} = x^2 \mathbf{i} + y^2 \mathbf{j} \)

### 17.9 Exercises

1-4 Verify that the Divergence Theorem is true for the vector field \( \mathbf{F} \) on the region \( E \).

1. \( \mathbf{F}(x, y, z) = 3x \mathbf{i} + xy \mathbf{j} + 2xz \mathbf{k} \),
   - \( E \) is the cube bounded by the planes \( x = 0, x = 1, y = 0, y = 1, z = 0, \) and \( z = 1 \)

2. \( \mathbf{F}(x, y, z) = x^2 \mathbf{i} + xy \mathbf{j} + z \mathbf{k} \),
   - \( E \) is the solid bounded by the paraboloid \( z = 4 - x^2 - y^2 \) and the \( xy \)-plane

3. \( \mathbf{F}(x, y, z) = xy \mathbf{i} + yz \mathbf{j} + xz \mathbf{k} \),
   - \( E \) is the solid cylinder \( x^2 + y^2 \leq 1, 0 \leq z \leq 1 \)

4. \( \mathbf{F}(x, y, z) = x \mathbf{i} + y \mathbf{j} + z \mathbf{k} \),
   - \( E \) is the unit ball \( x^2 + y^2 + z^2 \leq 1 \)

5-15 Use the Divergence Theorem to calculate the surface integral \( \iiint_S \mathbf{F} \cdot d\mathbf{S} \), that is, calculate the flux of \( \mathbf{F} \) across \( S \).

5. \( \mathbf{F}(x, y, z) = e^x \sin y \mathbf{i} + e^x \cos y \mathbf{j} + yz^2 \mathbf{k} \),
   - \( S \) is the surface of the box bounded by the planes \( x = 0, x = 1, y = 0, y = 1, z = 0, \) and \( z = 2 \)

6. \( \mathbf{F}(x, y, z) = x^2z^3 \mathbf{i} + 2xyz^2 \mathbf{j} + xz^4 \mathbf{k} \),
   - \( S \) is the surface of the box with vertices \((\pm 1, \pm 2, \pm 3)\)

7. \( \mathbf{F}(x, y, z) = 3xy^2 \mathbf{i} + xe^x \mathbf{j} + z^4 \mathbf{k} \),
   - \( S \) is the surface of the solid bounded by the cylinder \( y^2 + z^2 = 1 \) and the planes \( x = -1 \) and \( x = 2 \)

8. \( \mathbf{F}(x, y, z) = x^2y \mathbf{i} - x^2y^2 \mathbf{j} - x^3yz \mathbf{k} \),
   - \( S \) is the surface of the solid bounded by the hyperboloid \( x^2 + y^2 - z^2 = 1 \) and the planes \( z = -2 \) and \( z = 2 \)

9. \( \mathbf{F}(x, y, z) = xy \sin z \mathbf{i} + \cos(xz) \mathbf{j} + y \cos z \mathbf{k} \),
   - \( S \) is the ellipsoid \( x^2/a^2 + y^2/b^2 + z^2/c^2 = 1 \)

10. \( \mathbf{F}(x, y, z) = x^2 \mathbf{i} - x^2y \mathbf{j} + 2xy \mathbf{k} \),
    - \( S \) is the surface of the tetrahedron bounded by the planes \( x = 0, y = 0, z = 0, \) and \( x + 2y + z = 2 \)

11. \( \mathbf{F}(x, y, z) = (\cos z + xy^2) \mathbf{i} + xe^x \mathbf{j} + (\sin y + xz^2) \mathbf{k} \),
    - \( S \) is the surface of the solid bounded by the paraboloid \( z = x^2 + y^2 \) and the plane \( z = 4 \)

12. \( \mathbf{F}(x, y, z) = x^2 \mathbf{i} - x^2y \mathbf{j} + 4xyz \mathbf{k} \),
    - \( S \) is the surface of the solid bounded by the cylinder \( x^2 + y^2 = 1 \) and the planes \( z = x + 2 \) and \( z = 0 \)

13. \( \mathbf{F}(x, y, z) = 4x^2 \mathbf{i} + 4yz \mathbf{j} + 3z \mathbf{k} \),
    - \( S \) is the sphere with radius \( R \) and center the origin