Math 302 practice midterm 1

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Duration: 50 minutes.

Instructions:

- Write your name and student ID on every page.
- This examination contains five questions worth a total of 40 points.
- Write each answer very clearly below the corresponding question (Use back of page if needed).
- No calculators, books, notebooks or any other written materials are allowed.
- Good luck!
1. (10 pts) Find the arclength of the curve parametrized by \( \mathbf{r}(t) = (a \cos t \sin t, a \sin^2 t, bt) \), \( t \in [0,5] \) where \( a, b > 0 \).

\[
\mathbf{r}'(t) = (a(\cos^2 t - \sin^2 t), 2a \sin t \cos t, b)
\]

\[
|\mathbf{r}'(t)| = \sqrt{a^2(\cos^2 t + \sin^2 t) + b^2} = \sqrt{a^2 + b^2}
\]

\[
L = \int_{0}^{5} |\mathbf{r}'(t)| \, dt = \int_{0}^{5} \sqrt{a^2 + b^2} \, dt = 5 \sqrt{a^2 + b^2}
\]
2. (10pts) If a particle traverses the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) in the counterclockwise direction at constant speed 10, find the acceleration vector at the point \((a,0)\).

We first find the curvature at \((a,0)\).

For this we can use any parameterization of the ellipse. Let \(\mathbf{r}(t) = (a \cos t, b \sin t)\), \(t \in [0, \pi]\).

Note \(x(t)^2/a^2 + y(t)^2/b^2 = \cos^2 t + \sin^2 t = 1\), so this particle does trace out the ellipse.

\[
\mathbf{r}'(t) = (-a \sin t, b \cos t); \quad \mathbf{r}''(t) = (-a \cos t, -b \sin t)
\]

\[
\mathbf{r}''(t) \mathbf{r}'(t) = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
-a \sin t & b \cos t & 0 \\
-a \cos t & -b \sin t & 0
\end{vmatrix} = a b \mathbf{k}
\]

\[
\kappa(t) = \frac{|\mathbf{r}''(t) \mathbf{r}'(t)|}{|\mathbf{r}'(t)|^3} = \frac{a b}{(a^2 \sin^2 t + b^2 \cos^2 t)^{3/2}}.
\]

\[
\mathbf{r}'(0) = (a,0) \quad \text{so the curvature at } (a,0) = \kappa(0) = \frac{a b}{b^3} = \frac{a}{b^2}
\]

Now consider the particle following the ellipse at constant speed \(v(t) = 10\). The acceleration of this particle at \((a,0)\) is

\[
\mathbf{a} = v^2 \mathbf{T} + v^2 \kappa \mathbf{N}
\]

\[
= 100 \frac{a^3}{b^2} (-\mathbf{T}) \quad \text{since the principal normal at } (a,0) \text{ is } -\mathbf{T}, \text{ (115, 1.2)}
\]

\[
= \left(-100 \frac{a^3}{b^2}\right) \mathbf{T}
\]
3. (5 pts) State carefully: the definition of torsion at a point on a smooth curve.

Let \( \tilde{r}(s) \) be an arclength parametrization of the curve. The torsion \( \tau(s) \in \mathbb{R} \) is the real no. so that

\[
\frac{d}{ds} \frac{\tilde{T}(s)}{s} = -\tau(s) \tilde{N}(s).
\]

4. (5 pts) A comet follows the elliptical path shown below. The sun is at the origin. Describe and draw a rough sketch of the path traced out by the comet's velocity vector, where the tail of the velocity vector is at the origin.

By Hamilton's Thm. \( \tilde{V}(t) \) traces out a circle centered at \( \tilde{v}_0 \) with radius \( \frac{GM}{J} \) where \( J \) is the angular momentum of the comet, \( M \) = mass of sun, \( G \) = universal gravitation constant.
5. (10 pts) A particle moving in the $x - y$ plane starts at \( t = 0 \) from the point \( x = 1, y = 0 \). Let \((r(t), \theta(t))\) denote its polar coordinates at time \( t \). It is known that \( r(t) = (t + 1)^2 \), the speed of the particle at time \( t \) is \( v(t) = \sqrt{8(t + 1)} \), and that \( \theta'(t) \geq 0 \) for all \( t \).

(a) Find \( \theta(t) \) for all \( t \geq 0 \).

\[
\begin{align*}
\hat{r}(t) &= (\cos \theta, \sin \theta) \quad \hat{\theta}(t) = \frac{dr}{d\theta} = (-\sin \theta, \cos \theta) \\
\hat{r}(t) &= \frac{d}{dt} (\theta(t)) \frac{d\theta}{dt} \\
\hat{v}(t) &= \hat{r}(t) \hat{r}(t) + \hat{r}(t) \hat{\theta}(t) \hat{\theta}(t) \\
&= (2(t + 1)) \hat{r}(t) + (t + 1)^2 \hat{\theta}(t) \hat{\theta}(t)
\end{align*}
\]

\[
\begin{align*}
\left( \frac{d\theta}{dt} \right)^2 &= (2(t + 1))^2 + (t + 1)^2 \left( \frac{d\theta}{dt} \right)^2 \\
4(t + 1)^2 &= (t + 1)^2 \left( \frac{d\theta}{dt} \right)^2 \\
\Rightarrow \frac{d\theta}{dt} &= \frac{2(t + 1)}{(t + 1)^2} \\
\Rightarrow \theta(t) &= \int_0^t \frac{2}{u^2} du = \frac{2}{t} 
\end{align*}
\]

(b) Find the first time \( t > 0 \) when the particle lies on the \( x \) axis.

\[
\begin{align*}
r(t) &= 1 \quad \text{at} \ t, \text{so} \ \hat{r}(t) \text{will next hit the x-axis.} \\
\theta(t) &\text{nondecreasing} \quad \text{when} \ \theta(t) = \frac{\pi}{2} \\
\Rightarrow 2 \log(t + 1) &= \frac{\pi}{2} \\
\Rightarrow \log(t + 1) &= \frac{\pi}{4} \\
\Rightarrow t &= e^{\frac{\pi}{4} - 1}
\end{align*}
\]