Problem 1 (8 points): Consider the curve with the parameterization

\[ \mathbf{r}(t) = \left( t^2, \frac{4\sqrt{3}}{3} t^{3/2}, 2t \right). \]

1. Find the length of the curve from \( t = 0 \) to \( t = T \).

2. Reparameterize the curve with respect to arclength measured from the point \((0,0,0)\) in the direction of increasing \( t \).

\[
\begin{align*}
1. & \quad \mathbf{r}'(t) = \left( 2t, \frac{4\sqrt{3}}{3} t^{1/2}, 2 \right) = \left( 2t, \frac{2\sqrt{3}}{3} t^{1/2}, 2 \right) \\
\text{Length} & \quad = \int_0^T \sqrt{\left( 2t \right)^2 + \left( \frac{4\sqrt{3}}{3} t^{1/2} \right)^2 + 2^2} \, dt = \int_0^T \sqrt{4t^2 + \frac{16}{3} t + 4} \, dt = 4 \left( t + \frac{2}{3} \right) \Big|_0^T = 4 \left( T + \frac{2}{3} \right)
\end{align*}
\]

\[
\begin{align*}
2. & \quad s(t) = t^2 + 2t \\
\left( t^2 + 2t \right) - s \rightarrow 0 \\
b & \quad = -2 \pm \frac{\sqrt{4 + 4s}}{2} = \frac{-2 \pm 2\sqrt{1 + s}}{2} \quad \Rightarrow \quad t > 0 \\
r_0(t) & \quad = \left( \frac{2}{\sqrt{1 + s} - 1} \right) \left( 1 + \frac{2}{3} \sqrt{1 + s} - 1 \right) = \frac{2}{\sqrt{1 + s} - 1} \left( \frac{2}{3} \frac{2}{\sqrt{1 + s} - 1} \right)
\end{align*}
\]
Problem 2 (8 points): Consider the curve with the parameterization

\[
r(t) = \sin(t^2) \mathbf{i} + \cos(t^2) \mathbf{j} + t^2 \mathbf{k}
\]

where \( t > 0 \). Find the unit tangent vector \( \mathbf{T}(t) \), the principle unit normal vector \( \mathbf{N}(t) \), the curvature \( \kappa(t) \), and the tangential and normal components of acceleration \( a_T(t) \) and \( a_N(t) \).

\[
\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{2t}} \left( \cos(t^2), -\sin(t^2), 1 \right)
\]

\[
|\mathbf{r}'(t)| = 2t \sqrt{\cos(t^2)^2 + \sin(t^2)^2 + 1} = 2\sqrt{2} t = \|\mathbf{v}(t)\| = \text{Speed}
\]

\[
\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{1}{\sqrt{2}} \left( \cos(t^2), -\sin(t^2), 1 \right)
\]

\[
\mathbf{N}'(t) = \frac{1}{\sqrt{2}} \left( 2t \left(-\sin(t^2)\right), -2t \cos(t^2), 0 \right) = \sqrt{2} t \left(-\sin(t^2), -\cos(t^2), 0 \right)
\]

\[
|\mathbf{N}'(t)| = \sqrt{2} t
\]

\[
\mathbf{a}(t) = \mathbf{v}(t) = \mathbf{r}''(t) = 2\sqrt{2}
\]

\[
\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{1}{2}\left(2\sqrt{2} \right)^2 = 4 t^2
\]
Problem 3. (4 points each). In each problem below, select the correct answer; you do not need to show work; no partial credit will be given.

Let \( \mathbf{r}(t) \) be a vector valued function. Let \( r', r'', \) and \( r''' \) denote \( \frac{d\mathbf{r}}{dt}, \frac{d^2\mathbf{r}}{dt^2}, \) and \( \frac{d^3\mathbf{r}}{dt^3} \) respectively.

1. \( \frac{d}{dt} \left[ (\mathbf{r} \times \mathbf{r}') \cdot \mathbf{r}'' \right] \) is given by
   (a) \( (\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'' \)
   (b) \( (\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r} + (\mathbf{r} \times \mathbf{r}') \cdot \mathbf{r}'' \)
   (c) \( \mathbf{r} \cdot (\mathbf{r}' \times \mathbf{r}'') \)
   (d) \( 0 \)
   (e) None of the above.

2. \( \frac{d}{dt} |\mathbf{r}(t)| \) is given by:
   (a) \( |\mathbf{r}'(t)| \)
   (b) \( \frac{\mathbf{r} \cdot \mathbf{r}'}{|\mathbf{r}|} \)
   (c) \( 2\mathbf{r} \cdot \mathbf{r}' \)
   (d) \( 0 \)
   (e) None of the above.

\[
\frac{d}{dt} \left( (\mathbf{r} \times \mathbf{r}') \times \mathbf{r}'' \right) = \frac{d}{dt} \left[ \begin{vmatrix} \mathbf{r} & \mathbf{r}' & \mathbf{r}'' \\ \mathbf{r}' & \mathbf{r}'' & \mathbf{r} \\ \mathbf{r}'' & \mathbf{r} & \mathbf{r}' \\ \end{vmatrix} \right] = \det \left( \begin{array}{c} \mathbf{r} \\ \mathbf{r}' \\ \mathbf{r}'' \end{array} \right) = \det \left( \begin{array}{c} \mathbf{r} \\ \mathbf{r}' \\ \mathbf{r}'' \end{array} \right)
\]

NB \( \to \) (But we didn't discuss scalar triple products yet) 

\[
\frac{d}{dt} |\mathbf{r}(t)| = \frac{d}{dt} \left( \mathbf{r}(t) \cdot \mathbf{r}(t) \right)^{1/2} = \frac{1}{2} \left( \mathbf{r}(t) \cdot \mathbf{r}(t) \right)^{-1/2} \frac{d}{dt} \left( \mathbf{r}(t) \cdot \mathbf{r}(t) \right) = \frac{1}{2} \mathbf{r}(t) \cdot \mathbf{r}(t) = \mathbf{r}(t) \cdot \mathbf{r}(t)
\]

\( \therefore (b) \) is correct.
Problem 4 (8 points). In the above diagram, three curves are shown. Curve A is the graph of a function \( f(x) \). Curve B is the graph of the function \( \kappa_A(x) \) where \( \kappa_A \) is the curvature of curve A as a function of \( x \). Curve C is the graph of the function \( \kappa_B(x) \) where \( \kappa_B \) is the curvature of curve B as a function of \( x \). On the diagram, label the curves as A, B, and C. Provide reasoning for your choices in the space below.

B has an inflection pl. at \( x = 0.75 \) so \( \kappa_B(0.75) = 0 \). \( ^* \): C must be \( \kappa_B \).

A looks like its curvature is increasing for \( x \leq 0.75 \), say.

(A particle traversing at unit speed has \( \frac{1}{\kappa} \) projection)

\( \therefore \) B is only possible curvature of A.

\( \kappa_B \leq \kappa_A \)
Problem 5 (8 points). Say whether the following statements are true (T) or false (F). You may assume that all functions and vector fields are defined everywhere and have derivatives of all orders everywhere. You do not need to give reasons; this problem will be graded by answer only. You will get +1 for each correct answer, -1 for each wrong answer, and 0 for each no answer.

1. The curve with vector equation \( r(t) = (t^3, 2t^3, 3t^3) \) is a line. \( T \)

2. The derivative of a vector valued function is obtained by differentiating each component function. \( T \)

3. If \( u(t) \) and \( v(t) \) are vector valued functions, then

\[
\frac{d}{dt} [u(t) \times v(t)] = u'(t) \times v'(t).
\]

\( F \)

4. If \( T(t) \) is the unit tangent vector of a parameterized curve, then the curvature is \( \kappa = \frac{|dT/dt|}{|r'|}. \)

\( F \)

5. The osculating circle of a curve \( C \) at a point has the same tangent vector, normal vector, and curvature as \( C \) at that point. \( T \)

6. If \( r(t) \) is a path in three space with constant curvature \( \kappa \), then \( r(t) \) parameterizes part of a circle of radius \( 1/\kappa \). \( F \) (e.g., helix)

7. Let \( A(t) \) be the area swept out by the trajectory of a planet from time \( t_0 \) to time \( t \). Then \( \frac{dA}{dt} \) is constant. \( T \)

8. The orbit of a planet is an ellipse with the sun at the center of the ellipse. \( F \) (its at a focus)