• There are 10 questions worth a total of 26. Answers to the problems for 0 marks are included below. These questions should not be handed in but those marked for positive credit should be. All the questions below are on material you should know for the midterm. A few additional questions will be added to this HW next Thursday.

1. 0 marks 3.12

**Solution:** The contrapositive is very useful here. The contrapositive says: “If \( x \) is not even, then \( 7x + 5 \) is not odd”, or, in short, “if \( x \) is odd, then \( 7x + 5 \) is even”. So let us prove this statement. Since \( x \) is odd, \( x \) can be written as \( x = 2k + 1 \) for some integer \( k \). Then \( 7x + 5 = 7(2k + 1) + 5 = 14k + 12 = 2(7k + 6) \), and so it is even.

2. 6 marks 3.14

3. 0 marks 3.20

**Solution:** We need to consider two cases – what happens when \( n \) is odd and what happens when \( n \) is even.

*Proof.* Let \( n \in \mathbb{Z} \) and so either \( n \) is even or \( n \) is odd. Assume \( n \) is even, so \( n = 2k \) for some \( k \in \mathbb{Z} \). Thus \( n^3 - n = 8k^3 - 2k = 2(4k^3 - k) \). Since \( 4k^3 - k \in \mathbb{Z} \), it follows that \( n^3 - n \) is even.

Similarly, assume \( n \) is odd and so \( n = 2k + 1 \) for some \( k \in \mathbb{Z} \). Thus \( n^3 - n = (8k^3 + 12k^2 + 6k + 1) - (2k + 1) = 2(4k^3 + 6k^2 + 2k) \). Since \( 4k^3 + 6k^2 + 2k \in \mathbb{Z} \), it follows that \( n^3 - n \) is even.

Alternatively we could do a bit more work at the beginning and realise that \( n^3 - n = n(n-1)(n+1) \). A little more briefly we have

*Proof.* Let \( n \in \mathbb{Z} \). If \( n \) is even then \( n = 2k \) for some \( k \in \mathbb{Z} \) and so \( n^3 - n = 2k(2k-1)(2k+1) \) which is even. Similarly if \( n \) is odd then \( n = 2k + 1 \) for some \( k \in \mathbb{Z} \). Thus \( n^3 - n = 2k(2k+1)(2k+2) \) which is also even.

4. 0 marks 3.21 (Hint: Not sure this is best proved by Cases).

**Solution:** We prove the contrapositive. If \( x \) or \( y \) is even then \( xy \) is even. Without loss of generality assume \( x \) is even then \( x = 2k \) for some \( k \in \mathbb{Z} \) and so \( xy = 2ky \) which is even. Thus if \( x \) or \( y \) is even then \( xy \) is even.

5. 2 marks 3.22 (Hint: use 3.21)
6. **5 marks** 4.30

7. **3 marks** Prove that for any sets $A$ and $B$, $A \Delta B = \emptyset$ iff $A = B$. Here recall that $A \Delta B = (A - B) \cup (B - A)$.

8. **0 marks** Consider the statement: For any sets $A$ and $B$, $(A \cup B) - B = A$. If it is true, provide a proof. If it is false give a counter-example.

**Solution:** It is false. For example, take $A = \{1\}$, $B = \{1, 2\}$. Then $A \cup B = \{1, 2\}$, and $(A \cup B) - B = \emptyset$, but $A \neq \emptyset$.

9. **5 marks** (a) Find a counterexample to the statement: For all $x, y \in (0, \infty)$, $\sqrt{xy} \geq \frac{x+y}{2}$.
   (b) Prove the arithmetic-geometric mean inequality:
   
   $$\forall x, y \in [0, \infty), \sqrt{xy} \leq \frac{x+y}{2}.$$  

   You may use the
   
   **Lemma.** For all non-negative real numbers $r$ and $s$, $r \leq s \iff r^2 \leq s^2$.

10. **5 marks** (a) Find a counterexample to the statement: For all $x, y \in (0, \infty)$, $\sqrt{xy} \geq \frac{x+y}{2}$.
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    **Lemma.** For all non-negative real numbers $r$ and $s$, $r \leq s \iff r^2 \leq s^2$. 