There are 10 questions worth a total of 21. Not all questions may be graded.

1. **0 marks** Q2.2 f — 1 mark

   **Solution:**
   
   - (f) is false since 53 is prime and not equal to 2.

2. **3 marks** Q2.6 — 1 each.

   **Solution:** Since $A \in \mathcal{P} (\{1, 2, 4\})$, $A$ is a subset of $\{1, 2, 4\}$.
   
   - We need $A$ to be a subset of $\{1, 2, 3\}$, so we need $A = \emptyset, \{1\}, \{2\}$ or $\{1, 2\}$.
   - We need $A$ to not be a subset of $\{1, 2, 3\}$, so we need $4 \in A$. Thus $A = \{4\}, \{1, 4\}, \{2, 4\}$ or $\{1, 2, 4\}$.
   - We need $A$ to be disjoint from $\{1, 2, 3\}$, so we need $A = \emptyset$ or $\{4\}$.

3. **4 marks** Q2.14 — 1 each

   **Solution:** $P$ is false and $Q$ is true
   
   - "17 is odd" — $\neg P$ is true
   - "17 is even or 19 is prime" — $P \lor Q$ is $F \lor T$ which is true
   - "17 is even and 19 is prime" — $P \land Q$ is $F \land T$ which is false
   - "if 17 is even then 19 is prime" — $P \Rightarrow Q$ is $F \Rightarrow T$ which is true.

4. **0 marks** Q2.18 — 1 each

   **Solution:**
   
   - If $5n + 3$ is prime then $7n + 1$ is prime.
   - If 13 is prime then 15 is prime — $T \Rightarrow F$ is false.
   - If 33 is prime then 43 is prime — $F \Rightarrow T$ is true.

5. **4 marks** Q2.20(a,b)
Solution: The statement \( P \Rightarrow Q \) is false when \( P \) is true but \( Q \) is false; otherwise it is true.

- \( P \) is true when \( x = 7 \) and \( Q \) is false when \( x < 8 \). So \( P \Rightarrow Q \) is false when \( x = 7 \) and true for all \( x \in \mathbb{R} \setminus 7 \).
- \( P \) is true when \( x \geq 1 \) or \( x \leq -1 \) and \( Q \) is false when \( x < 1 \). Thus \( P \Rightarrow Q \) is false when \( x \leq -1 \) and true when \( x > -1 \).

6. **2 marks** Write out the truth tables for \( P \Rightarrow Q \) and \( Q \Rightarrow P \). For what truth values of \( P \) and \( Q \) do the the truth values of \( P \Rightarrow Q \) and \( Q \Rightarrow P \) differ?

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<th>( P )</th>
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Solution: The truth values differ when \( P \) is T and \( Q \) is F, or \( P \) is F and \( Q \) is T.

7. **3 marks** Consider the statement: I will play hockey only if you pay me 5 million next year and buy me a house in the British Properties.

(i) Write out the above in the form, if \( P \), then \( Q \), for appropriate \( P \) and \( Q \).

Solution: Introduce the following statements:
\( H \): I play hockey; \( M \): You pay me $5 million;
\( P \): you buy me a house in the British Properties
\( H \Rightarrow M \land P \), or equivalently, if \( H \), then \( M \land P \).

(ii) Write out the converse and contrapositive, first in logical symbols, and then in clear English (state it as simply as possible).

Converse: \( M \land P \Rightarrow H \). If you pay me $5 million and buy me a house in the British Properties, then I will play hockey.
Contrapositive: \( \neg(M \land P) \Rightarrow \neg H \) or equivalently \( \neg M \lor \neg P \Rightarrow \neg H \).
If you don’t pay me $5 million or don’t buy me a house in the British Properties, then I won’t play hockey.

8. **0 marks** 2.38 First, use the fact that logical equivalence means the biconditional is true rather than writing out the truth tables. For establishing the biconditional recall that \( A \Rightarrow B \) is T providing that you can establish that \( B \) is T when \( A \) is T. Now verify the above by checking that the truth tables are the same.
Solution:
Solution 1.
We will prove the biconditional

\[(P \lor Q) \Rightarrow R \iff (P \Rightarrow R) \land (Q \Rightarrow R)\] is T. \hspace{1cm} (1)

First assume

\[(P \lor Q) \Rightarrow R. \hspace{1cm} (2)\]

We must show that \(P \Rightarrow R\) and \(Q \Rightarrow R\).
Assume \(P\) is T. Then so is \(P \lor Q\) and hence by (2), so is \(R\). This proves \(P \Rightarrow R\) and the same reasoning shows that \(Q \Rightarrow R\). We have shown that \(\Rightarrow\) holds in (1).

Next assume

\[(P \Rightarrow R) \land (Q \Rightarrow R). \hspace{1cm} (3)\]

We must show \((P \lor Q) \Rightarrow R\), so assume \(P \lor Q\). This means \(P\) is T or \(Q\) is T, so assume \(P\) is T. By (3) we have \(P \Rightarrow R\) and so \(R\) is T. Similarly if \(Q\) is T, we can show that \(R\) is T. We have proved that \((P \lor Q) \Rightarrow R\), as required. This shows that \(\Leftarrow\) holds in (1) and the proof is complete.

Well that was exhausting but it is what most mathematicians would have done. Here is a simpler

Solution 2. Using logical equivalence

\[
(P \lor Q) \Rightarrow R \equiv (\sim (P \lor Q)) \lor R
\]

\[
\equiv (\sim P \land \sim Q) \lor R \hspace{1cm} \text{DeMorgan}
\]

\[
\equiv (\sim P \lor R) \land (\sim Q \lor R) \hspace{1cm} \text{Distributive}
\]

\[
\equiv (P \Rightarrow R) \land (Q \Rightarrow R)
\]

Finally we give the proof by Truth Table.
Solution 3.

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9. **2 marks** 2.40
Solution:

- \( \neg(P \lor Q) \equiv \neg P \land \neg Q \): Both \( x \) and \( y \) are non-zero.
- \( \neg(P \land Q) \equiv \neg P \lor \neg Q \): Either \( a \) is odd or \( b \) is odd.

10. **3 marks** Consider the statement: The fish are biting and there are no bugs, or the fish are not biting and there are bugs, or it is winter.

Write out the negation of the above in English. You should simplify your answer as much as possible, being sure of course that it is logically equivalent to the negation. Justify your answer.

**Solution:** \( P \): the fish are biting; \( B \): there are bugs; \( W \): it is winter.

So we want \( \neg \left( (P \land \neg B) \lor (\neg P \land B) \lor W \right) \). Using a double application of De Morgan’s laws this is logically equivalent to

\[
\neg(P \land \neg B) \land (\neg(P \land B)) \land \neg W
\]

\[
\equiv ((\neg P) \lor B) \land (P \lor \neg B) \land \neg W. \quad (4)
\]

This answer is not bad but one can simplify further using the distributive laws. A double application of the distributive laws given in the text (we did this version in class in Sec. 202) shows that

\[
(R \lor S) \land (T \lor U) \equiv (R \land T) \lor (R \land U) \lor (S \land T) \lor (S \land U).
\]

Use this in (4), noting that \( P \land \neg P \) and \( B \land \neg B \) are always \( F \) to see that (4) is logically equivalent to

\[
\left( F \lor (\neg P \land \neg B) \lor (B \land P) \lor F \right) \land (\neg W)
\]

\[
\equiv \left( \neg P \land \neg B \lor (B \land P) \right) \land (\neg W)
\]

\[
\equiv (\neg P \land \neg B \land \neg W) \lor (P \land B \land \neg W).
\]

This reads: the fish are not biting and there are no bugs and it is not winter, or the fish are biting and there are bugs and it is not winter.