Math 217 Midterm 2

Instructor: Prof. Ed Perkins

Duration: 50 minutes.

Instructions:

• Write your name on every page, and student ID on page 1.
• This examination contains four questions with total weight of 50 points.
• Write each answer clearly below the corresponding question (Use back of opposite page if needed).
• No calculators, books, notebooks or any other written materials are allowed.
• Good luck!
1. (12 pts) Evaluate: \( \iint_D e^{y^3} \, dA \), where \( D \) is the bounded region in the first quadrant \((x \geq 0, y \geq 0)\) between \( y = \sqrt{x} \) and \( y = 1 \).

\[
\begin{align*}
\iint_D e^{y^3} \, dA &= \int_0^1 \int_0^{\sqrt{x}} e^{y^3} \, dy \, dx \\
&= \int_0^1 y^2 e^{y^3} \, dy \quad \left( u = y^3, \quad \frac{dy}{3} = y^2 \, dy \right) \\
&= \int_0^1 e^{u \frac{1}{3}} \, du \\
&= \frac{e - 1}{3}
\end{align*}
\]
2. (18 pts) Let \( f(x, y, z) = x^2 + y^2 + z^2 - x - 2y - 2z \) on its domain 
\[ D = \{(x, y, z) : x^2 + y^2 + z^2 \leq 9\} \]

(a) Explain why \( f \) has absolute minimum and absolute maximum values on its domain.

Dislosed and bounded, \( f \) is continuous on \( D \). By the Fund Thm of Extreme Values, \( f \) attains its abs. max. and abs. min. values.

(b) Find all the critical points of \( f \).

\[ \nabla f(x, y, z) = (2x - 1, 2y - 2, 2z - 2) = \mathbf{0} \]
\[ \Rightarrow \quad x = \frac{1}{2}, \quad y = 1, \quad z = 1 \]
\[ \Rightarrow (\frac{1}{2}, 1, 1) \text{ is only c. p.} \]

(c) Find the absolute minimum and absolute maximum values of \( f \) and all points where they are attained.

\[ f(\frac{1}{2}, 1, 1) = \frac{1}{4} + 1 + 1 - \frac{1}{2} - 2 = 2 \cdot \frac{1}{2} = -\frac{1}{2} \]

\[ \partial f = \{(x, y) : x^2 + y^2 + z^2 = 9, z \geq 0\} \]

To find abs max. and abs min.

\[ \nabla f(x, y, z) = \lambda \nabla g(x, y, z) \]
\[ \Rightarrow (x, y, z) = \lambda (2x, 2y, 2z) \]
\[ \Rightarrow (-1, -2, -2) = \lambda (2x_1, 2y_1, 2z_1) \]
\[ \Rightarrow x_1 = \frac{1}{2}, \quad y_1 = -\frac{1}{2}, \quad z_1 = 1 \]
\[ \Rightarrow (x_1, y_1, z_1) \text{ by } (1) - (2) \]
\[ \Rightarrow 2x = y \]

\[ \Rightarrow x^2 + y^2 + z^2 = 9 \Rightarrow x^2 + 2x^2 + z^2 = 9 \Rightarrow x = \pm 1 \]
\[ \Rightarrow (x, y, z) = (1, 1, 2) \text{ or } (-1, -2, -2) \]
\[ f(1, 1, 2) = 1 + 1 + 4 + 4 = 10 \]
\[ f(-1, -2, -2) = 1 + (-1 - 2) = 1 + 1 + 4 = 6 \]
\[ \text{Abs. max. value is } 10 \text{ at } (-1, -2, -2); \text{ abs. min. value is } -2\frac{1}{2} = \frac{-5}{2}. \]
3. (15 pts) Let \( T(x, y, z) = e^{-x^2 - 2y^2 - 3z^2} \) be the temperature at \((x, y, z)\).

(a) If a particle is at \((1, 1, 1)\), in what direction should it head to warm up as quickly as possible?

\[
\nabla T(1,1,1) = (-2x, -4y, -6z) e^{-x^2 - 2y^2 - 3z^2}
\]

\[
\frac{\nabla T(1,1,1)}{\left| \nabla T(1,1,1) \right|} = \left( \frac{-2}{\sqrt{11}} \right)
\]

(b) Suppose another particle is passing through \((1, 1, 1)\) with (instantaneous) velocity \((e^5, 2e^5, -2e^5)\), what is the rate of change of its (*) temperature? (You may assume the particle's position vector \(r(t)\) has a continuous derivative.)

(*) that is, the temperature it feels.

Let \( \vec{r}(t)\) = position of particle where it passes through \((1,1,1)\) at \(t=0\):

\[
r'(0) = (e^5, 1, 3, -2)
\]

\[
\frac{d}{dt} T(\vec{r}(t)) \bigg|_{t=0} = \nabla T(\vec{r}(0)). \vec{r}'(0) \quad \text{(Chain Rule)}
\]

\[
= \nabla T(1,1,1). (e^5, 1, 3, -2)
\]

\[
= (-2, -4, -6) e^{-6} \cdot e^5 (1, 3, -2)
\]

\[
= e^{-1} [-2 - 8 + 12]
\]

\[
= 2e^{-1}
\]
4. (5 pts) Suppose $f$ is a differentiable function on $\mathbb{R}^2$. It is known that $D_uf(0, 0) < D_if(0, 0)$ for all unit vectors $u \neq i$. Find $\frac{\partial f}{\partial y}(0, 0)$.

\[ \frac{\partial f}{\partial y}(0, 0) = 0 \] (equal second components of above vectors)