\textbf{Math 217\ HW 6\ Solutions}\n
12.9 \#38. \( f(x,y) = -x^2 - y^2 + \sqrt{3}x - y - 1 \) on \( R = \{ (x,y) : x^2 + y^2 \leq 2 \} \).

\( f \) is continuous; \( R \) is a solid disk, hence closed and bounded.

\( f \) has abs. min. and abs. max. values by the F.T.E.V.

i. \( C \subset D \) \( \Rightarrow \) \( \Phi (x,y) = (-2x + \sqrt{3}, -2y - 1) = \Phi (0,0) \)

\( \Rightarrow \) \( x = \frac{\sqrt{3}}{2}, \ y = -\frac{1}{2} \) \( \Rightarrow \) \( (x,y) = (\frac{\sqrt{3}}{2}, -\frac{1}{2}) \)

\( f \left( \frac{\sqrt{3}}{2}, -\frac{1}{2} \right) = -\frac{3}{4} - \frac{1}{2} + \frac{3}{4} + \frac{1}{2} - 1 = 0. \)

12.) \( \partial R = \{ (x,y) : x^2 + y^2 = 2 \} \) circle of radius \( \sqrt{2} \) centered at \((0,0)\).

Let \( g(\theta) = f(\sqrt{2} \cos \theta, \sqrt{2} \sin \theta) \) \( , \ \theta \in [0, 2\pi) \).

(abs max and min of \( f \) on \( \partial R \))

\( g(0) = -6 + \frac{\sqrt{2}}{2} (\sqrt{2} \cos 0) - \sqrt{2} \sin 0 - 1 = -6 + \frac{\sqrt{2}}{2} \cdot 2 - 1 = -\frac{11}{2} \)

\( g(\frac{\pi}{2}) = -6 + \frac{\sqrt{2}}{2} (\sqrt{2} \cos \frac{\pi}{2}) - \sqrt{2} \sin \frac{\pi}{2} - 1 = -6 + \frac{\sqrt{2}}{2} \cdot 0 - 1 = -7 \)

\( g(\pi) = -6 + \frac{\sqrt{2}}{2} (\sqrt{2} \cos \pi) - \sqrt{2} \sin \pi - 1 = -6 + \frac{\sqrt{2}}{2} \cdot (-2) - 1 = -6 - \sqrt{2} \)

\( g(\frac{3\pi}{2}) = -6 + \frac{\sqrt{2}}{2} (\sqrt{2} \cos \frac{3\pi}{2}) - \sqrt{2} \sin \frac{3\pi}{2} - 1 = -6 + \frac{\sqrt{2}}{2} \cdot 0 - 1 = -7 \)

\( g(2\pi) = -6 + \frac{\sqrt{2}}{2} (\sqrt{2} \cos 2\pi) - \sqrt{2} \sin 2\pi - 1 = -6 + \frac{\sqrt{2}}{2} \cdot 2 - 1 = -\frac{11}{2} \)

Ends; \( g(0) = g(2\pi) = 3 \sqrt{2} - 7 \)

\( \Rightarrow \) Abs max of \( f \) is \( 3 \sqrt{2} - 7 \); Abs min of \( f \) is \( -2 \sqrt{2} - 7 \)

13.) So Abs max value of \( f \) is \( 3 \sqrt{2} - 7 \) (at \( P_{\text{max}} = \left( \frac{\sqrt{3}}{2}, -\frac{1}{2} \right) \) - not required)

Abs min value of \( f \) is \( -2 \sqrt{2} - 7 \) (at \( P_{\text{min}} = \left( -\frac{\sqrt{3}}{2}, \frac{\sqrt{2}}{2} \right) \))

\( \Rightarrow \) \( \frac{1}{2} \left( -3 \sqrt{2}, \sqrt{6} \right) \)
12.4 # 24. \( f(x, y, z) = (x-1)^2 + (y-2)^2 + z^2 \) in \( \mathbb{R}^3 \).

Let \( S = \{(x, y, z) \mid z^2 - x^2 - y^2 = 0 \} \cap \mathbb{R}^3 \).

If \( f \) has an \( \text{abs min} \) on \( S \), \((x-1)^2 + (y-2)^2 + z^2 \leq 16 \) (closed and bounded) by \( \text{FTBV} \). Hence, as discussed in class \( f \) has an \( \text{abs min} \).

Let \((x_1, y_1, z_1)\) be point where \( \text{abs min} \) is achieved.

\[ \frac{\partial f}{\partial (x, y, z)} = \lambda \frac{\partial g(x, y, z)}{\partial (x, y, z)} \quad \text{for some} \quad \lambda \in \mathbb{R}^3 \]

\(
(2x_1-1, 2y_1-2, 2z_1) = (-2x_1 - 2y_1, 2z_1) = (-4x_1, -4y_1, 2z_1)
\)

1) \(2x_1 = -2x \) \quad 2) \(2y_1 = -2y \) \quad 3) \(2z_1 = 2z \)

\[ x_1 = 1 \quad \text{or} \quad \lambda = 0 \]

Case 1 \( \lambda = 0 \Rightarrow x_1 = y_1 = 0 \) by 4) \( z_1^2 = x_1^2 + y_1^2 \)

\[ f(0, 0, 0) = 5 \]

Case 2 \( \lambda = 1 \quad \text{and} \quad \lambda \neq 0 \)

\[ \partial f \quad \text{become} \quad \partial f' \quad \text{at} \quad (x_1, y_1) = \left( -\frac{x_1}{2} \right) \quad \text{and} \quad \lambda = 1 \left( \begin{array}{l}
2x_1 = -4x_1 \\
y_1 = 4 \\
z_1 = \frac{1}{2}
\end{array} \right)
\]

\[ (x_1, y_1) = \left( \frac{1}{2}, 1 \right) \]

so by 4) \( z_1^2 = \frac{1}{4} + \frac{1}{2} = \frac{3}{4} \)

\[ z_1 = \frac{1}{2} \left( \sqrt{3} \right) \frac{1}{2} \]

\[ f(\frac{1}{2}, 1, \frac{1}{2}) = \left( \frac{1}{2} - 1 \right)^2 + (1 - 2)^2 + \left( \frac{1}{2} \right)^2 = \frac{1}{4} + 1 + \frac{1}{4} = \frac{3}{2} \]

The points on \( S \) closest to \( (1, 2, 0) \) are \((\frac{3}{2}, 1, \sqrt{\frac{3}{2}})\) and \((\frac{1}{2}, 1, -\sqrt{\frac{3}{2}})\)
12.9 #52(b) Let $D = \mathbb{R}^n$ be open, $f, g \in C^1(D)$, and $S = \{ (x_1, \ldots, x_n) \in D : g(x_1, \ldots, x_n) = 0 \}$.   

If $P$ is a local extreme of $f|_{S}$ and $\nabla g(P) \neq 0$, then there is a real no. $\lambda$ so that $\nabla f(P) = \lambda \nabla g(P)$. 

Let $f(x_1, \ldots, x_n) = \frac{1}{2} \sum_{i=1}^{n} x_i^2$, $D = \{ (x_1, \ldots, x_n) : x_1 > 0, \ldots, x_n > 0 \}$, $g(x_1, \ldots, x_n) = x_1 \cdots x_n$, $S_K = \{ (x_1, \ldots, x_n) \in D : x_1 \cdots x_n = k \}$. 

$S_K$ is closed and bounded so by FBBV $f|_{S_K}$ has an absolute max value. Clearly it will not occur when $x_i = 0$ and so it will occur at some $P_{\text{max}} \in S_K$. 

To find it we use the above formulation of Lagrange Multipliers in $n$ dimensions, 

$$( \nabla f(x_1, \ldots, x_n) )_i = \frac{\partial}{\partial x_i} f(x_1, \ldots, x_n) , \quad \nabla g(\xi_1, x_2, \ldots, x_n) = (1, 1, \ldots, 1)$$ 

2nd component 

$$( \nabla f(x_1, \ldots, x_n) ) = \lambda \nabla g(x_1, \ldots, x_n) = (\lambda, \lambda, \ldots, \lambda)$$ 

$$( \frac{\partial}{\partial x_i} f(x_1, \ldots, x_n) ) = \lambda \frac{\partial}{\partial x_i} g(x_1, \ldots, x_n) = \lambda$$ 

As $x_i > 0 \text{ at any max}$, $\lambda > 0$ and so $x_i = \frac{\pi}{\lambda} \frac{x_i}{\lambda}$ all $i$. 

$\therefore x_1 = \ldots = x_n$. Since $P_{\text{max}} \in S_K$ we get $x_1 = k/n \text{ all } i$, 

$$( f(x_1, \ldots, x_n) ) = ( f(1, x_2, \ldots, x_n) ) = (\frac{k}{n})^n \text{ must be the abs. max of } f|_{S_K}$$
\[ x_i \in \left( \frac{k}{n} \right)^n = \left( \frac{\ell x_i}{n} \right)^n \quad \text{on all } \ell > 0. \]

\[ \left( x_i \right)^n \leq \frac{\ell x_i}{n} \quad \text{for all } x_i > 0, n > 0 \]

\[ \left( x_i \right)^n \leq \frac{x_i}{n} \quad \text{for all } i, j \geq 0 \]

because if any \( x_j = 0 \), the above inequality is obvious.

**3.**

17.9 #54. Minimize \( f(x, y, z) = x^2 + y^2 + z^2 \) on \[ L: \begin{align*}
1: & \quad x + 2z - 12 = 0 \\
2: & \quad x + y - 1 = 0 \\
3: & \quad y = x \\
4: & \quad y = x \\
5: & \quad y = x \\
6: & \quad y = x \\
7: & \quad y = x \\
8: & \quad y = x \\
9: & \quad y = x
\end{align*} \] A nearest point \( P_{min} \) on \( L \cap \{ x^2 + y^2 + z^2 \leq 10^{10} \} = B \) exists because \( f \) is continuous and \( B \) is closed and bounded.

\[ f \] has a minimum since \( P_{min} \) will be closer to \( B \) than any other point in \( L \setminus \{ x^2 + y^2 + z^2 \leq 10^{10} \} \) as these points have \( f > 10^{10} \).

To find \( P_{min} \) by #53, we solve:

\[ \nabla f = \lambda \nabla g + \mu \nabla h \]

\[ \begin{cases}
L: & \quad (2x, 2y, 2z) = \lambda (1, 0, 2) + \mu (1, 1, 0) = (\lambda + 1, \mu + 0, 2 \lambda) \\
M: & \quad 2x = \lambda + 1 \quad \Rightarrow \quad \lambda = 2x - 1 \\
N: & \quad 2y = \mu \quad \Rightarrow \quad \mu = 2y \\
O: & \quad 2z = 2 \lambda \quad \Rightarrow \quad z = \lambda.
\end{cases} \]

Put \( \lambda \) and \( \mu \) into L:

\[ \begin{align*}
\text{Put (5) and (6) into (4):} & \quad 6 - \frac{2x}{3} = 2(2 - 16 - x) = 4x - 12 \\
& \quad 7.18 = 9.72 \Rightarrow x = 4 \quad \text{and so } y = 6 - x = 2 \quad \text{and} \quad z = 6 - \frac{2x}{3} = 4.
\end{align*} \]

\[ \therefore (14, 3, 4) \text{ must be nearest point to } B \text{ on } L. \]
4. 13.2 #38

**Problem:**

It's easier to integrate with \( x \) first. (Why?)

\[
\text{Sl. } x^2y \, dA = \int_0^2 \int_{y^2}^{\sqrt{4-y^2}} x^2 y \, dx \, dy = \int_0^2 y \left[ \frac{(2-y)^3}{3} - \frac{y^6}{3} \right] \, dy
\]

\[
= \frac{1}{3} \left[ y^4 + 6y^3 + 3y^2 + 8y - \frac{y^7}{7} \right]_0^2
\]

\[
= \frac{1}{3} \left[ 2^4 + \frac{2^6}{4} + \frac{12 \cdot 2^3}{3} + \frac{8 \cdot 2^2}{2} - \frac{2^7}{7} \right] = \frac{\frac{232}{7}}{3}
\]
13.2 #7b

\[ y = \frac{4}{3} x \]

\[ a^2 + \left( \frac{4}{3} a \right)^2 = 1 \]

\[ \Rightarrow a = \left( \frac{112}{57} \right) \]

\[ SS xy \ dA = \int_{-\frac{3\sqrt{2}}{2}}^{\frac{3\sqrt{2}}{2}} \int_{-\frac{3\sqrt{2}-2x^2}{3}}^{\frac{3\sqrt{2}-2x^2}{3}} xy \ dy \ dx \]

\[ = \int_{-\frac{3\sqrt{2}}{2}}^{\frac{3\sqrt{2}}{2}} \left[ \frac{x^3}{2} \right]_{-a}^{a} \ dx \]

\[ = \int_{-\frac{3\sqrt{2}}{2}}^{\frac{3\sqrt{2}}{2}} \left( \frac{16}{18} - 18x + \frac{72}{2} \right) \ dx \]

\[ = \left[ 8 - 9x^2 + 72x + 9x^3 \right]_{-\frac{3\sqrt{2}}{2}}^{\frac{3\sqrt{2}}{2}} \]

\[ = 0 \]

\[ \therefore \int_{R} xy \ dA = 0 \]

by symmetry of \( R_1 \) about the \( y \) axis.

so \( xy \) cancels with \( \int yx \) when \( R_2 \) is used.

\[ \int_{R} xy \ dA = 0 \] by symmetry of \( R_2 \) about the \( y \) axis.

so \( xy \) cancels with \( x - y \).

\[ \therefore \int_{R} xy \ dA = \int_{R_1} xy \ dA + \int_{R_2} xy \ dA = 0 \]
Problem 5

(a) \( p, q > 0 \), \( \frac{1}{p} + \frac{1}{q} = 1 \) \( \implies \frac{q}{p} = 1 \).

\( f(x, y) = \frac{x^p}{y^q} \).
\[ D = \{ (x, y) : x > 0, y > 0 \} \]
\[ f \in C^1(D) \]

\( g(x, y) = xy \) \( g \in C^1(D) \).
Let \( C = \{ (x, y) : xy = c \} \) \( c > 0 \).

Assume \( f \mid C \) has an abs. min at \((x_0, y_0)\). (may assume it exists)

By Theorem Lagrange multipliers, there is a \( \lambda \in \mathbb{R} \) s.t.

\[ \nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0) \]
\[ \implies \left( x_0, y_0^{-1} \right) = \lambda (y_0, x_0) \]
\[ \implies x_0^{p-1} = xy \]
\[ \implies y_0^{q-1} = x_0 \]
\[ \implies \frac{x_0}{y_0} = \frac{y_0^{q-1}}{x_0} \]
\[ \implies \lambda = \frac{y_0^{q-1}}{x_0} \]

(b) \( f(x_0, y_0) = (\frac{c^{2/p+1}}{p}) + (\frac{c^{1/q+1}}{q}) \)
\[ = \left( \frac{1}{p} \right) c^{2p/q+1} = C \]

(b) If \( a > 0 \) or \( b > 0 \), \( ab > 0 \) and the conclusion is obvious.

Assume \( a, b > 0 \) and let \( c = ab > 0 \).

By (a)
\[ \frac{a^p}{p} + \frac{b^q}{q} > C = ab \].