11.2 Vectors in Three Dimensions

47. **Submarine course** A submarine climbs at an angle of 30° above the horizontal with a heading to the northeast. If its speed is 20 knots, find the components of the velocity in the east, north, and vertical directions.

48. **Maintaining equilibrium** An object is acted upon by the forces \( \mathbf{F}_1 = (10, 6, 3) \) and \( \mathbf{F}_2 = (0, 4, 9) \). Find the force \( \mathbf{F}_3 \) that must act on the object so that the sum of the forces is zero.

Further Explorations

49. **Identifying sets** Give a geometric description of the following sets.

\[ x^2 + x^2 + 2y - 4z - 4 = 0 \]
\[ x^2 + x^2 + 6x + 6y - 8z - 2 = 0 \]
\[ x^2 - 14y + z^2 \geq -13 \]
\[ x^2 - 14y + z^2 \leq 63 \]
\[ x^2 + x^2 - 8x + 14y - 18z \geq 65 \]
\[ x^2 + x^2 - 8x + 14y - 18z \leq 65 \]

**Vector operations** For the given vectors \( \mathbf{u} \) and \( \mathbf{v} \), evaluate the expressions.

\[ 3\mathbf{u} + 2\mathbf{v} \quad \mathbf{u} - \mathbf{v} \quad |\mathbf{u} + 3\mathbf{v}| \]

\[ = (1, 3, 0), \mathbf{v} = (3, 0, 2) \]
\[ = (-1, 1, 0), \mathbf{v} = (2, -4, 1) \]
\[ = (-7, 5, 1), \mathbf{v} = (-2, 4, 0) \]
\[ = (5, 1, 3\sqrt{2}), \mathbf{v} = (2, 0, 7\sqrt{2}) \]

**Unit vectors and magnitude** Consider the following points.

\[ \mathbf{a} \cdot \mathbf{b} \cdot \mathbf{c} \]

Find \( \mathbf{Q} \) and state your answer in two forms: \( (a, b, c) \) and \( a\mathbf{i} + b\mathbf{j} + c\mathbf{k} \).

Find the magnitude of \( \mathbf{Q} \).

Find two unit vectors parallel to \( \mathbf{Q} \).

50-52. **Sets of points** Describe with a sketch the sets of points \((x, y, z)\) satisfying the following equations.

\[ (x + 1)(y - 3) = 0 \]
\[ x^2y^2z^2 > 0 \]
\[ y - z = 0 \]

53-56. **Parallel vectors of varying lengths** Find vectors parallel to \( \mathbf{v} \) of the given length.

\[ \mathbf{v} = (6, -8, 0); \text{length: 20} \]
\[ \mathbf{v} = (3, -2, 6); \text{length: 10} \]
\[ \mathbf{v} = \mathbf{PQ} \text{ with } P(3, 4, 0) \text{ and } Q(2, 3, 7); \text{length: 3} \]
\[ \mathbf{v} = \mathbf{PQ} \text{ with } P(1, 0, 1) \text{ and } Q(2, -1, 1); \text{length: 3} \]

57. **Collinear points** Determine whether the points \( P, Q, \text{ and } R \) are collinear (lie on a line) by comparing \( \mathbf{PQ} \) and \( \mathbf{PR} \). If the points are collinear, determine which point lies between the other two points.

\[ a. \ P(1, 6, -5), Q(2, 5, -3), R(4, 3, 1) \]
\[ b. \ P(1, 5, 7), Q(5, 13, -1), R(0, 3, 9) \]
\[ c. \ P(1, 2, 3), Q(2, -3, 6), R(3, -1, 9) \]
\[ d. \ P(9, 5, 1), Q(11, 18, 4), R(6, 3, 0) \]

58. **Collinear points** Determine the values of \( x \) and \( y \) such that the points \((1, 2, 3), (4, 7, 1), \text{ and } (x, y, 2)\) are collinear (lie on a line).

59. **Lengths of the diagonals of a box** A fisherman wants to know if his fly rod will fit in a rectangular 2 ft \( \times 3 \) ft \( \times 4 \) ft packing box. What is the longest rod that fits in this box?

**Applications**

60. **Forces on an inclined plane** An object on an inclined plane does not slide provided the component of the object’s weight parallel to the plane \( |\mathbf{W}_{\text{perp}}| \) is less than or equal to the magnitude of the opposing frictional force \( |\mathbf{F}_f| \). The magnitude of the frictional force, in turn, is proportional to the component of the object’s weight perpendicular to the plane \( |\mathbf{W}_{\text{perp}}| \) (see figure). The constant of proportionality is the coefficient of static friction, \( \mu \).
SECTION 11.3 EXERCISES

Review Questions
1. Define the dot product of \( \mathbf{u} \) and \( \mathbf{v} \) in terms of their magnitudes and the angle between them.
2. Define the dot product of \( \mathbf{u} \) and \( \mathbf{v} \) in terms of the components of the vectors.
3. Compute \((2, 3, -6) \cdot (1, -8, 3)\).
4. What is the dot product of two orthogonal vectors?
5. Explain how to find the angle between two nonzero vectors.
6. Use a sketch to illustrate the projection of \( \mathbf{u} \) onto \( \mathbf{v} \).
7. Use a sketch to illustrate the scalar component of \( \mathbf{u} \) in the direction of \( \mathbf{v} \).
8. Explain how the work done by a force in moving an object is computed using dot products.

Basic Skills
9-12. Dot product from the definition Consider the following vectors \( \mathbf{u} \) and \( \mathbf{v} \). Sketch the vectors, find the angle between the vectors, and compute the dot product using the definition \( \mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}|| ||\mathbf{v}|| \cos \theta \).
9. \( \mathbf{u} = 4\mathbf{i} \) and \( \mathbf{v} = 6\mathbf{j} \)
10. \( \mathbf{u} = (-3, 2, 0) \) and \( \mathbf{v} = (0, 0, 6) \)
11. \( \mathbf{u} = (10, 0) \) and \( \mathbf{v} = (10, 10) \)
12. \( \mathbf{u} = (-\sqrt{3}, 1) \) and \( \mathbf{v} = (\sqrt{3}, 1) \)

13-18. Dot products and angles Compute the dot product of the vectors \( \mathbf{u} \) and \( \mathbf{v} \), and find the approximate angle between the vectors.
13. \( \mathbf{u} = 4\mathbf{i} + 3\mathbf{j} \) and \( \mathbf{v} = 4\mathbf{i} - 6\mathbf{j} \)
14. \( \mathbf{u} = (3, 4, 0) \) and \( \mathbf{v} = (0, 4, 5) \)
15. \( \mathbf{u} = (-10, 0, 4) \) and \( \mathbf{v} = (1, 2, 3) \)
16. \( \mathbf{u} = (3, -5, 2) \) and \( \mathbf{v} = (-9, 5, 1) \)
17. \( \mathbf{u} = 2\mathbf{i} - 3\mathbf{k} \) and \( \mathbf{v} = \mathbf{i} + 4\mathbf{j} + 2\mathbf{k} \)
18. \( \mathbf{u} = \mathbf{i} - 4\mathbf{j} - 6\mathbf{k} \) and \( \mathbf{v} = 2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k} \)

19-22. Sketching orthogonal projections Find \( \text{proj}_\mathbf{u} \) and \( \text{scal}_\mathbf{u} \) by inspection without using formulas.
19. 

20-28. Calculating orthogonal projections For the given vectors \( \mathbf{u} \) and \( \mathbf{v} \), calculate \( \text{proj}_\mathbf{u} \) and \( \text{scal}_\mathbf{u} \).
20. \( \mathbf{u} = (-1, 4) \) and \( \mathbf{v} = (4, 2) \)
21. \( \mathbf{u} = (10, 5) \) and \( \mathbf{v} = (2, 6) \)
22. \( \mathbf{u} = (-8, 0, 2) \) and \( \mathbf{v} = (1, 3, -3) \)
23. \( \mathbf{u} = (3, -5, 2) \) and \( \mathbf{v} = (-9, 5, 1) \)
24. \( \mathbf{u} = 2\mathbf{i} - 4\mathbf{k} \) and \( \mathbf{v} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \)
25. \( \mathbf{u} = \mathbf{i} + 4\mathbf{j} + 7\mathbf{k} \) and \( \mathbf{v} = 2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k} \)
3.2. Normal vectors

Find a vector normal to the given vectors.

1. (1, 2, 3) and (−2, 0, 3)
2. (0, 1, 2) and (−3, 0, 4)
3. (0, 0, 4) and (−8, 2, 1)
4. (6, −2, 4) and (1, 2, 3)

4.4 Cross Products

46-49. Areas of triangles

Find the area of the following triangles T. (The area of a triangle is half the area of the corresponding parallelogram.)

46. The sides of T are \( u = (0, 6, 0) \), \( v = (4, 4, 4) \), and \( u - v \).
47. The sides of T are \( u = (3, 3, 3) \), \( v = (6, 0, 6) \), and \( u - v \).
48. The vertices of T are \( O(0, 0, 0) \), \( P(2, 4, 6) \), and \( Q(5, 7) \).
49. The vertices of T are \( O(0, 0, 0) \), \( P(1, 2, 3) \), and \( Q(5, 4) \).

50. A unit cross product

Under what conditions is \( u \times v \) a unit vector?

51. Vector equation

Find all vectors \( u \) that satisfy the equation \( (1, 1, 1) \times u = (-1, -1, 2) \).

52. Vector equation

Find all vectors \( u \) that satisfy the equation \( (1, 1, 1) \times u = (0, 0, 1) \).

53. Area of a triangle

Find the area of the triangle with vertices on the coordinate axes at the points \( (c, 0, 0) \), \( (0, c, 0) \), and \( (0, 0, c) \), in terms of \( a \), \( b \), and \( c \).

54-56. Scalar triple product

Another operation with vectors is the scalar triple product, defined to be \( u \cdot (v \times w) \), for vectors \( u \), \( v \), and \( w \) in \( \mathbb{R}^3 \).

54. Express \( u \), \( v \), and \( w \) in terms of their components and show that \( u \cdot (v \times w) \) equals the determinant

\[
\begin{vmatrix}
  u_1 & u_2 & u_3 \\
  v_1 & v_2 & v_3 \\
  w_1 & w_2 & w_3 \\
\end{vmatrix}
\]

55. Consider the parallelepiped (slanted box) determined by the position vectors \( u \), \( v \), and \( w \) (see figure). Show that the volume of the parallelepiped is \( |u \cdot (v \times w)| \).

56. Prove that \( u \cdot (v \times w) = (u \times v) \cdot w \).

Applications

57. Bicycle brakes

A set of caliper brakes exerts a force on the rim of a bicycle wheel that creates a frictional force \( F \) of 40 N.