1. (8 marks) Write down the trapezoidal Rule and Simpson’s Rule approximations \( T_4 \) and \( S_4 \) for \( I = \int_0^2 e^{-4x^2} \, dx \). You may express your answers as sums involving exponential terms.

\[
\Delta x = \frac{b-a}{4} = \frac{1}{2}
\]

\[
T_4 = \Delta x \left( \frac{f(0)}{2} + f(1) + f(2) + f(3) + \frac{f(2)}{2} \right)
\]

\[
= \frac{1}{2} \left( \frac{1}{2} + e^{-1} + e^{-4} + e^{-9} + e^{-16} \right)
\]

\[
S_4 = \frac{\Delta x}{3} \left( f(0) + 4f(1) + 2f(2) + 4f(3) + f(2) \right)
\]

\[
= \frac{1}{6} \left( 1 + 4e^{-1} + 2e^{-4} + 4e^{-9} + e^{-16} \right)
\]
2. (13 marks) Let $R$ be the (bounded) region that lies between the curves $y = e^{-x^2/2}$, $y = 0$, $x = 0$ and $x = 1$.

a) (3 marks) Give a rough sketch of the region $R$. Clearly show the four “corner” points of $R$.

b) (5 marks) Find the volume of the solid obtained by rotating $R$ about the $y$-axis

\[
V = \int_{0}^{1} 2\pi x e^{-x^2/2} \, dx
\]

\[
= 2\pi \int_{0}^{\sqrt{2}} e^{-u} \, du
\]

\[
= 2\pi \left[ 1 - e^{-\sqrt{2}/2} \right]
\]

c) (5 marks) A particle starts at the origin and moves around the boundary of $R$ in the clockwise direction until it returns to its starting point. Express the total distance the particle travels in terms of a definite integral. Do not evaluate the integral.

Distance

\[
= l_1 d_0 + l_1 d_1 + l_2 d_2 + l_2 d_3
\]

\[
= 1 + \int_{0}^{1} \sqrt{1 + 2x^2 e^{-x^2}} \, dx + e^{-\sqrt{2}/2} + 1
\]

\[
= 2 + e^{-\sqrt{2}/2} + \int_{0}^{1} \sqrt{1 + x^2 e^{-x^2}} \, dx
\]
3. (9 marks)

Find all $p > 0$ so that the improper integral $\int_0^{\pi/2} (\sin x)^{-1/2} x^{-p} \, dx$ is finite.

(You may use any results we proved in class, if you state them carefully.)

We showed in class:

\[ y = \sin x \leq \frac{2}{\pi} x \leq \frac{3}{\pi} x, \quad \forall x \in [0, \pi/2] \]

1) $x^p \sqrt{\frac{\pi}{2}} x^{-1/2} \geq \int_0^{\pi/2} (\sin x)^{-1/2} x^{-p} \, dx \geq x^{-1/2-p} \quad \forall x \in [0, \pi/2]$.

By Comp. Test for Improper S'IS:

\[
\int_0^{\pi/2} x^{-p} \sin x^{-1/2} \, dx < \infty \quad \Rightarrow \quad \int_0^{\pi/2} x^{-p-1/2} \, dx < \infty \quad \Rightarrow \quad \int_0^{\pi/2} x^{-p} (\sin x)^{-1/2} \, dx < \infty.
\]

\[
\int_0^{\pi/2} (\sin x)^{-1/2} x^{-p} \, dx < \infty \quad \Rightarrow \quad \int_0^{\pi/2} x^{1/2-p} \, dx < \infty
\]

\[
\Rightarrow \quad \frac{1}{2} + p < 1
\]

\[
\Rightarrow \quad p < \frac{1}{2}
\]
4. (10 marks) Prove the following assertions. As always, you may use (without proof) all results proven in class, but you should provide proofs for results used from the Practice Midterm.

a) (5 marks) Let \( \{a_n\} \) be a bounded sequence, that is, for some \( R \) we have \( |a_n| \leq R \) for all \( n \). If \( \bar{a}_n = \sup\{a_k : k \geq n\} \), then \( \{\bar{a}_n\} \) is convergent.

\[
\bar{a}_n \geq a_k \quad \forall k \geq n \Rightarrow \bar{a}_n \geq a_k \quad \forall k \geq n+1
\]

\( \bar{a}_n \) is an upper bound for \( S = \{a_k : k \geq n\} \) and \( \bar{a}_n \) is the least upper bound of \( S \). \( \bar{a}_n \geq \sup S = \bar{a}_{n+1} \). \( \Rightarrow \) \( \bar{a}_n \) is decreasing

\( \bar{a}_n \geq a_n \geq -R \Rightarrow \) \( \bar{a}_n \) is bounded below by \( -R \)

\( \Rightarrow \) and \( \Rightarrow \) \( \bar{a}_n \) is convergent (Decreasing Sequence Dichotomy)

b) (5 marks) If \( \{a_n\} \) is an increasing sequence such that \( \{a_{10n}\} \) is convergent, then \( \{a_n\} \) is convergent.

\[
\min_{m,n} 10m \leq 10n \Rightarrow a_{10n} = a_{10m}.
\]

\( \{a_{10n}\} \) is decreasing. So \( \{a_{10n}\} \) converges. \( \{a_{10n}\} \) is bounded above

\[ \exists R \text{ st. } \forall n \in \mathbb{N} \quad a_{10n} \leq R \]

\[ \forall n \in \mathbb{N} \quad a_n = a_{10n} \text{ since } n \leq 10 \text{ and } a_{10n} \text{ is increasing} \]

\[ a_n \leq R \]

\( \Rightarrow \) \( \{a_n\} \) is increasing and bounded above.

\( \Rightarrow \) \( \{a_n\} \) is convergent (Increasing Sequence Dichotomy)