1. (18 marks) Evaluate the following integrals:
   
   a) (4 marks) \[ \int \frac{\cos x}{\sin x + 2} \, dx \]
   
   Let \( u = \sin x \), \( du = \cos x \, dx \)
   
   \[ I = \int \frac{1}{u+2} \, du = \log |\sin x + 2| + C \]
   
   b) (7 marks) \[ \int_{0}^{1} \frac{dx}{(4-x^2)^{3/2}} \]
   
   Simplify your answer completely.
   
   Let \( u = 2 \sin \theta \), \( dx = 2 \cos \theta \, d\theta \), \( 2 \sin \frac{\pi}{6} = \frac{1}{2} \)
   
   \[ I = \int_{0}^{\pi/6} \frac{2 \cos \theta}{(2 \cos \theta)^3} \, d\theta \]
   
   \[ = \frac{1}{16} \int_{0}^{\pi/6} \sec^{4} \theta \, d\theta \]
   
   \[ = \frac{1}{16} \int_{0}^{\pi/6} (1 + \tan^2 \theta) \sec^2 \theta \, d\theta \]
   
   \[ u = \tan \theta, \quad du = \sec^2 \theta \, d\theta \]
   
   \[ = \frac{1}{16} \int_{0}^{\pi/6} (1 + u^2) \, du \]
   
   \[ = \left. \frac{1}{16} \left( u + \frac{u^3}{3} \right) \right|_{0}^{\pi/6} \]
   
   \[ = \frac{1}{16} \left( \frac{\sqrt{3}}{2} + \frac{1}{3 \cdot 3} \right) = \frac{10}{16 \cdot \sqrt{3}} \]
   
   \[ = \frac{5 \sqrt{3}}{216} \]

   Question 1 continued on page 2 ...
1. (continued)

c) (7 marks) \[ \int \frac{2x-x^2-1}{x^2(1+x^2)} \, dx \]

\[
\frac{2x-x^2-1}{x^2(1+x^2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C + D}{1+x^2}.
\]

\[-x^2+2x-1 = A\cdot2(1+x^2) + B(1+x^2) + x^2(C + D)\]

\[
\begin{cases}
  x=0 \Rightarrow B = -1 \\
  x=0 \Rightarrow x^2 + 2x - 1 = \frac{A+C}{D} x^3 + (B+D) x^2 + \left(\frac{A}{2}\right) x + (l-1)
\end{cases}
\]

\[
A = 2, \quad C = -2, \quad -1 + D = -1 \Rightarrow D = 0
\]

\[
\therefore I = \int \frac{2}{x} - \frac{1}{x^2} - \frac{2}{1+x^2} \, dy
\]

\[
= 2 \log |x| + \frac{1}{x} - \int \frac{du}{u}, \quad u = 1+x^2, \quad du = 2xdx
\]

\[
= 2 \log |x| + \frac{1}{x} - \log (1+x^2) + c
\]
2. (8 marks) True or False. If True provide a short proof using any results from the course or assignments. If False, provide, and justify, a counter-example.

(a) Any continuous function \( f : [a, b] \to [0, \infty) \) has an antiderivative.

\[ \text{True.} \]

Let \( F(x) = \int_a^x f(t) \, dt \). (exists in \( [a, b] \), \( F(a) = 0 \))

By the FTC: \( F'(x) = f(x) \)

\( F \) is an antiderivative of \( f \).

(b) Let \( [x] \) denote the greatest integer less than or equal to \( x \). Then \( f(x) = ([x] - [x^2]) + e^{10x} \) is integrable on \([0, 8]\).

\[ \text{True} \]

\( g(x) = [x] \) is increasing, hence integrable on \([0, 8]\)

\( h(x) = [x^2] \) is increasing on \([0, 8]\)

\( f(x) = e^{10x} \) is continuous, hence integrable on \([0, 8]\)

\( f(x) = g(x) - h(x) + f(x) \) is integrable by

Linearity of the Integral
3. (8 marks) For any positive integer \( n \), let \( P_n \) be the partition of \([0, 1]\) consisting of\n\( x_i = \frac{i^2}{n^2}, \ i = 0, 1, \ldots, n \) (\( P_n \) has \( n \) intervals that are not of equal length).

a) (2 marks) Find the norm of \( P_n \) (the maximum width of a subinterval) and show that it approaches 0 as \( n \to \infty \).

\[
\| P_n \| = \max_{1 \leq i \leq n} \frac{(i+1)^2}{n^2} - \frac{i^2}{n^2} = \max_{1 \leq i \leq n} \frac{2i+1}{n^2} = \frac{2n+1}{n^2} = \frac{2}{n} - \frac{1}{n^2} \\
\to 0 \quad \text{as} \quad n \to \infty.
\]

b) (6 marks) By computing a limit of Riemann sums for the partitions \( P_n \), evaluate \( \int_0^1 x^{1/2} \, dx \) without using the Fundamental Theorem of Calculus. You may use the formulas

\[
\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \quad \text{and} \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}
\]

Let \( C_i = x_i \in [x_{i-1}, x_i] \). If \( f(x) = x^{1/2} \), then \( f(C_i) = x_i^{1/2} \).

\[
R_\epsilon(f, P_n, C) = \sum_{i=1}^{n} x_i \left( \frac{1}{2} \left( x_i - x_{i-1} \right) \right) = \sum_{i=1}^{n} i \left( \frac{1}{n} \left( \frac{2i-1}{n^2} \right) \right) = \frac{1}{n} \sum_{i=1}^{n} \left( \frac{2i-1}{n^2} \right) = \frac{1}{n^3} \sum_{i=1}^{n} (2i - 1)
\]

\[
= \frac{1}{n^3} \left[ \frac{2n(n+1)}{2} - \frac{n(n+1)}{2} \right] \quad \text{(by above formula)}
\]

\[
= \frac{(n+1)(n-1)}{2n^2} - \frac{1}{2n^2}
\]

As \( f \) is continuous and \( \| P_n \| \to 0 \) we have

\[
\int_0^1 x^{1/2} \, dx = \lim_{n \to \infty} R_\epsilon(f, P, C) = \lim_{n \to \infty} \frac{1}{n} \left( \frac{2n(n+1)}{2} - \frac{n(n+1)}{2} \right) - \frac{1}{2n^2}
\]

\[
= \frac{1}{3} - 0 = \frac{2}{3}.
\]
4. (6 marks)

a) (2 marks) Let \( f(x) = \int_{-\infty}^{\infty} \sqrt{t^2 + 1} \, dt \). Find \( f''(x) \).

Let \( F(y) = \int_{0}^{y} \sqrt{t^2 + 1} \, dt \), then \( F'(y) = \sqrt{y^2 + 1} \) \( \left( r = r_2 \right) \)

So \( F(y) = F(y^2) - F(x) \)

\( \therefore f'(x) = F'(x^2) - 2x - F'(x) \)

\( = \left[ \sqrt{x^2 + 1} \right] 2x - \sqrt{x^2 + 1} \quad \text{dy h} \)

b) (4 marks) Let \( h \) be a continuous function on \([-1, 1]\). Prove that \( h \) is an odd function if and only if \( \int_{-x}^{x} h(u) \, du = 0 \) for all \( x \in (0, 1] \).

\((\Rightarrow) \) Let \( h(u) = -h(-u) \). Let \( w = -u \), \( dw = -du \)

\( \therefore \int_{-x}^{x} h(u) \, du = -\int_{-x}^{x} h(-w) \, dw = \int_{x}^{-x} h(-w) \, dw = \int_{-x}^{x} h(w) \, dw \)

\( \therefore \int_{-x}^{x} h(w) \, dw = 0 \quad \forall x \in [0, 1] \)

\((\Leftarrow) \) Assume \( \int_{-x}^{x} h(w) \, dw = 0 \quad \forall x \in [0, 1] \quad (x = 0 \text{ is trivial})

Differentiate and use the FT,

\( h(x) - (-h(-x)) = 0 \)

\( \therefore h(x) + h(-x) = 0 \)

\( \therefore h(-x) = -h(x) \quad \forall x \in [-1, 1] \)

\( \therefore h \text{ is odd} \)