21 marks

1. **Short Problems.** Each question is worth 3 points. Put your answer in the box provided and show your work. No credit will be given for the answer without the correct accompanying work. Simplify your answers as much as possible.

(i) Find the average of \( f(x) = \frac{2\log x}{x} \) on \([1,e]\).

\[
\overline{f} = \frac{1}{e-1} \int_1^e \frac{2\log x}{x} \, dx = \frac{1}{e-1} \int_0^\infty 2u \, du = \frac{1}{e-1}
\]

Answer: \( \frac{1}{e-1} \)

(ii) Evaluate \( \int \cos^4 x \, dx \).

\[
\begin{align*}
\int \cos^4 x \, dx &= \int \left( \frac{1 + \cos 2x}{2} \right)^2 \, dx \\
&= \frac{1}{4} \int 1 + 2\cos 2x + \cos^2 2x \, dx \\
&= \frac{1}{4} \left[ x + \sin 2x + \frac{1 + \cos 4x}{2} \, dx \right] \\
&= \frac{1}{4} \left[ x + \sin 2x + \frac{x}{2} + \frac{1}{8} \sin 4x \right] + C \\
&= \frac{3}{8} x + \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + C
\end{align*}
\]

Answer: \( \frac{3}{8} x + \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + C \)
(iii) Calculate the following limit: \( \lim_{x \to 0} \frac{\arctan x - x + \frac{x^3}{3}}{2x^5} \). As always, justify your answer.

\[
= \lim_{x \to 0} \frac{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} - x + \frac{x^3}{3}}{2x^5} \\
= \lim_{x \to 0} \frac{\sum_{n=2}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}}{2x^5} \\
= \lim_{x \to 0} \frac{\sum_{n=2}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}}{2x^5} \\
= \frac{1}{10} \text{ by continuity of the power series}
\]

(iv) Calculate the definite integral \( \int_{-1}^{0} \frac{1}{(3 - 2x - x^2)^{3/2}} \, dx \).

\[
= \int_{-1}^{0} \frac{1}{(4 - (x+1)^2)^{3/2}} \, dx \\
= \int_{0}^{\frac{1}{2}} \frac{1}{(4 - u^2)^{3/2}} \, du \\
= \frac{2 \csc \theta}{2} \, du \\
= \frac{\sqrt{3}}{2} \cos^3 \theta \\
= \frac{1}{4} \int_{0}^{\frac{1}{2}} \frac{\sqrt{3}}{2} \cos^3 \theta \\
= \frac{1}{4} \tan \frac{\theta}{2} \\
= \frac{1}{4} \cdot \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{12}
\]
(v) Calculate the sum of the following series 
\[ S = \sum_{n=3}^{\infty} \left( \frac{4^n - 2}{4n - 3} + \log \left( \frac{3n}{4n - 3} \right) - \log \left( \frac{3n + 3}{4n + 1} \right) \right) \]

\[ S = \frac{1}{2} \sum_{n=3}^{\infty} \left( \frac{1}{2^n} \right)^2 + \lim_{n \to \infty} \sum_{n=3}^{\infty} \left[ \log (3n) - \log (3n + 3) \right] - \log (3n + 3) - \log (4n + 1) \] 

\[ = \frac{1}{2} \sum_{n=3}^{\infty} \left( \frac{1}{2^n} \right)^2 + \lim_{n \to \infty} \left[ \log 9 - \log 3(4n+1) - \log 9 + \log (4n+1) \right] \] 

\[ = \frac{1}{2} \sum_{n=3}^{\infty} \left( \frac{1}{2^n} \right)^2 + \lim_{n \to \infty} \log \left( \frac{4n+1}{3n+3} \right) \] 

\[ = \frac{4}{5} + \log \left( \frac{4}{9} \right) \hspace{1cm} \text{using the continuity of } \log \text{ at } \frac{4}{9} \] 

\[ \text{and } \frac{4n+1}{3n+3} \to \frac{4}{9} \]

(vi) Find an integral which corresponds to \( \lim_{n \to \infty} \sum_{i=1}^{n} \frac{4}{n} e^{-4x^2} \cos \left( 4 + \frac{4i}{n} \right)^2 \). You need not evaluate the integral.

\( P_n = \{ x_0, x_1, \ldots, x_n \} \) is a partition of \([x, y]\).

\( L(f) = \sum_{i=1}^{n} \frac{4}{n} e^{-4x^2} \cos \left( 4 + \frac{4i}{n} \right)^2 \).

\( x_i = x_{i-1} + \frac{y-x}{n} \), \( \Delta x = \frac{y-x}{n} \). \( \Delta x = x_i - x_{i-1} \). 

\[ \int_{x}^{y} e^{-4x^2} \cos \left( 4 + \frac{4i}{n} \right)^2 dx \]

\( f \) \( \Delta x \)

\[ \rightarrow \int_{x}^{y} f(x) \, dx \]

\text{by} \ \text{only} \ \text{of} \ \text{f}
(vii) Determine the radius of convergence and the interval of convergence of the power series
\[ \sum_{n=1}^{\infty} a_n x^n, \] where \( a_n = 2019 + (-1)^n. \)

**Answer:**

\[ \frac{1}{2}, \left( -1, 1 \right) \]

1. \[ a_{2n} = 2020, \quad a_{2n+1} = 2018. \]
2. \[ \left| a_{2(n+1)} \right| = \left| 2019 \right| \cdot \frac{1}{2} n \quad \rightarrow \quad 1 \]
3. \[ \frac{1}{R} = \lim_{n \to \infty} a_n^{-1/n} = 1 \quad \Rightarrow \quad R = 1 \]

If \( x = 1, \) then \( \left| a_n x^n \right| = \left| a_n \right| \geq 2019 \) by

\[ \frac{a_n}{2} \cdot a_n x^n \not\rightarrow 0 \]
\[ \frac{a_n}{2} \cdot a_n x^n \text{ is divergent if } x = 1. \]

\[ \text{set of converges } \left( -1, 1 \right) \]
2. Suppose that \( y(x) \) satisfies the differential equation

\[
\frac{dy}{dx} = -\frac{y^2 + y}{x + 1}, \quad \text{with } y(0) = -\frac{1}{2}.
\]

Find \( y(x) \) for all \( x \geq 0 \).

\[ 2 \geq -1 \]

\[ \int \frac{dy}{y^2 + y} = - \int \frac{1}{x + 1} \, dx \]

\[ \Rightarrow \int \frac{1}{y} - \frac{1}{y+1} \, dy = -\log(x+1) + C \]

Solve until \( y \notin (-1, 0) \).

\[ \Rightarrow \log\left(\frac{y}{y+1}\right) = C - \log(x+1) \]

\[ \Rightarrow \left| \frac{y}{y+1} \right| = e^{c_1} \quad (C_1 > 0) \]

\[ x=0 \quad \left| \frac{-\frac{1}{2}}{\frac{1}{2}} \right| \cdot C_1 : C_1 = 1 \]

\[ \left| \frac{y}{y+1} \right| = x+1 \]

At \( x=0 \) \[ \frac{y}{y+1} = \frac{-\frac{1}{2}}{\frac{1}{2}} < 0 \]

\[ \Rightarrow \frac{y}{y+1} = -(x+1) \quad \text{until } y \notin (-1, 0) \quad \frac{y}{y+1} < 0 \]

\[ \frac{1}{y} = -2-x \]

\[ y = -\frac{1}{2+x} \quad C(-1,0) \quad x \geq 0 \]

above analysis holds \( \forall x \geq 0 \)

\[ y(1) = -\frac{1}{2.5} \]

\[ y(0) = -\frac{1}{2.5} \]
3. For each of the following series, decide if they converge absolutely, converge conditionally or diverge. Justify your answer.

(a) \( \sum_{n=1}^{\infty} \arctan(n) \frac{1}{3n^3 + 1} \)

\[
\left| \frac{\arctan n}{3n^3 + 1} \right| \leq \frac{\pi/2}{3n^3} = \frac{\pi}{6n^3} < \frac{1}{n^3}
\]

\( \frac{1}{n} \) diverges.

Thus, by the Comparison Test, \( \sum \frac{\arctan n}{3n^3 + 1} \) is absolutely convergent.

Answer: Absolutely convergent.

(b) \( \sum_{n=1}^{\infty} \frac{(-1)^n n^3}{2n^2 - 5n^3} \)

\[
\left| \frac{1-n^3}{2n^2 - 5n^3} \right| = \frac{1}{2n^2 - 5n^3} \to \frac{1}{5}
\]

\( \frac{1}{n^2} \) diverges.

Thus, terms do not approach 0 so series diverges.

Answer: Diverges.

(c) \( \sum_{n=3}^{\infty} \frac{\cos(n\pi) \log n}{n} \)

\[
= \sum_{n=3}^{\infty} \frac{(-1)^n \left( \log n \right)}{n}
\]

\[
\frac{d}{dx} \log x = \frac{1}{x} \log x < 0 \quad \forall x > e
\]

\( b = 1 \).

By Limit Comparison Test

By Alternating Series Test series converges.

Answer: Convergent.
4 marks

4. Determine the mass of the bounded region between the two curves $x = 1 + y^2$ and $x = 2(2 - y^2)$ if the density at $(x, y)$ is given by $\rho(x, y) = |y|$.

Answer: $\frac{3}{2}$
5 marks
5. A particle spirals into the origin along the polar graph \( r = \theta^{-p} \) for \( \theta \in [2\pi, \infty) \), where \( p > 0 \). Find all \( p > 0 \) so that the total distance travelled is finite. Here you may use any reasonable definition of "total distance travelled" but state what it is. You should justify your answer carefully.

\[
\text{Distance Traveled} = \lim_{n \to \infty} \int_{\theta \in [2\pi, \infty)} \sqrt{(f(\theta))^2 + \left( \frac{df}{d\theta} \right)^2} \, d\theta
\]

\[
= \lim_{n \to \infty} \int_{\theta \in [2\pi, \infty)} \sqrt{\theta^{-2p} + p^2 \theta^{-2p+2}} \, d\theta
\]

\[
\leq \int_{2\pi}^{\infty} \theta^{-p} \, d\theta = \frac{\theta^{1-p}}{1-p} \bigg|_{2\pi}^{\infty} = \frac{1}{1-p} \theta_{2\pi}^{1-p}
\]

By comparison, this is a convergent integral for \( p > 1 \).

Distance Traveled \( < \infty \) \( \iff \int_{2\pi}^{\infty} \theta^{-p} \, d\theta < \infty \)

\[\iff p > 1\]
6. Evaluate the following:

(i) \( \lim_{n \to \infty} \sum_{k=n+1}^{2n} \frac{1}{k} \). (Your evaluation should include a justification that the limit exists.)

\[
= \lim_{n \to \infty} \sum_{k=n+1}^{2n} \frac{1}{k} \approx \ln(2) \approx 0.6931 \quad (\text{by definition of a Riemann sum on } [0, 1])
\]

\[
= \lim_{n \to \infty} \sum_{k=n+1}^{2n} \frac{1}{k} \approx \ln(2) \approx 0.6931 \quad (\text{as the sum approximates the area under the curve of } y = \frac{1}{x} \text{ from } n+1 \text{ to } 2n)
\]

\[
= \lim_{n \to \infty} \sum_{k=n+1}^{2n} \frac{1}{k} \approx \ln(2) \approx 0.6931 \quad (\text{as the sum approaches the definite integral from } n+1 \text{ to } 2n)
\]

\[
= \lim_{n \to \infty} \sum_{k=n+1}^{2n} \frac{1}{k} = \ln(2) = 0.6931 \quad (\text{by the definition of a Riemann sum on } [1/2, 1])
\]

(ii) \( F'(1) \) where \( F(x) = \int_0^x 2e^{u^2} \, du \) and \( I(x) = \int_1^x e^{-t^2/2} \, dt. \)

\[
L = \frac{y}{2} \quad \text{and} \quad 2e^{-y^2/2} \quad \text{by (i)}
\]

\[
G'(x) = 2e^{x^2} \quad \text{and} \quad I(x) = e^{-x^2/2} \quad (\text{by (i)})
\]

\[
F'(x) = 2e^{x^2} \quad \text{and} \quad F'(1) = 2e^{1^2} = 2e \quad (\text{by (ii)}).
\]

\[
F'(1) = 2e^{1^2} = 2e \quad (\text{by (ii)}).
\]
4 marks

(iii) \( \lim_{n \to \infty} a_n \), where \( a_1 = 3 \) and \( a_{n+1} = \sqrt{2a_n - 1} \) for all \( n \in \mathbb{N} \). (Your evaluation should include a justification that the limit exists.)

**Answer:**

Let \( f(x) = \sqrt{2x - 1} \). \( f \) is mon on \((\frac{3}{2}, \infty)\)

\[
(x, y) = f((x, y))
\]

\[
2 = 2x - 1 \quad \Rightarrow \quad x = \frac{3}{2}
\]

Claim \( \{a_n\} \) \( a_n > \frac{3}{2} \), \( a_n \geq a_{n+1} \) \( \forall n \in \mathbb{N} \)

\( n=1 \quad a_1 = 3 \geq \frac{3}{2} \), \( a_2 = \sqrt{2 \cdot 3 - 1} = \sqrt{5} < 3 = a_1 \)

Assume \( \{a_n\} \): \( a_{n+1} = f(a_n) \leq f(a_{n-1}) \Rightarrow \frac{3}{2} \geq a_{n+1} \geq a_{n-1} \)

\( \therefore a_{n+2} = f(a_{n+1}) \leq f(a_n) \leq a_{n-1} \leq a_n \) by \( \{a_n\} \) and \( f \)

\( \therefore \{a_n\} \) holds.

By induction \( \{a_n\} \) holds \( \forall n \).

1. \( \{a_n\} \) is \( V \) and bounded below by \( \frac{3}{2} \)

2. \( L = \lim_{n \to \infty} a_n \) exists in \( \mathbb{R} \)

\( \therefore \) by continuity of \( f \): \( a_{n+1} = f(a_n) \)

\[
\Rightarrow L = \lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} f(a_n)
\]

\( = f(\lim_{n \to \infty} a_n) = \frac{3}{2} \)

By \( (1) \), \( L = 1 \) \( \Rightarrow \quad \text{an} \to 1 \)
7. Give examples of the following and briefly justify them (complete proofs are not needed). Grading scheme for each is two points for a correct example and two points for the brief justification:

4 marks

(i) A $C^\infty$ function on $\mathbb{R}$ which is not equal to its Maclaurin series for all $x \neq 0$.

\[ f(x) = \begin{cases} e^{-1/x^2} & x > 0 \\ 0 & x = 0 \end{cases} \]

\[ f^{(n)}(x) = 0 \quad \forall x \in \mathbb{R}^+ \quad \text{(discussed in class)} \]

\[ f \text{ is trivially infinitely differentiable at all other points} \]

Maclaurin series is \( \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \equiv 0 \)

\[ f(0) = 0 \quad \forall x > 0 \quad \therefore f(0) \neq 0 \quad \frac{d}{dx} \left( \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \right) \bigg|_{x=0} 

4 marks

(ii) A function \( f : [-1, 1] \to \mathbb{R} \) which is integrable over \([-1, x]\) for all \(-1 < x \leq 1\) but for which

\[ \frac{d}{dx} \int_{-1}^{x} f(t) \, dt \neq f(0). \]

\[ f(x) = \begin{cases} 1 & x = 0 \\ 0 & x > 0 \end{cases} \]

we show in class \( f_0 \) is Riemann integrable over any \([a, b]\) and \( \int_{a}^{b} f_0 \, dx = 0. \)

\[ f(0) = \int_{-1}^{0} f(x) \, dx = 0 \quad \forall \]

\[ \frac{d}{dx} f(0) = 0 \quad \forall \]

\[ \frac{d}{dx} f(0) = 0 \neq 1 = f(0) \quad b = f(0) \]
(iii) A bounded function $f : [0, 1] \rightarrow \mathbb{R}$ which is not Riemann integrable but such that $f^2$ is Riemann integrable on $[0, 1]$.

Let $g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0, 1] \\ 0 & \text{if } x \notin \mathbb{Q} \cap [0, 1] \end{cases}$.

So we have \( g = \frac{f^2}{2} \) on $[0, 1]$. If $f$ were Riemann integrable on $[0, 1]$, then so is $g = \frac{f^2}{2}$. We showed in class that $g$ is not Riemann integrable on $[0, 1]$. Therefore, $f$ cannot be Riemann integrable on $[0, 1]$. But $f^2 \equiv 1$ which is trivially integrable on $[0, 1]$ (e.g. by $\Pi$).
8. (i) Assume \( f : (1, \infty) \to \mathbb{R} \) is an increasing function. Prove that

\[
\sum_{k=2}^{n} f(k) \geq \int_{1}^{n} f(x) \, dx \geq \sum_{k=1}^{n-1} f(k).
\]

Let \( g(x) = f(k) \), \( k \in \mathbb{R} \), \( k \in \mathbb{N} \) such that

\[
g(k) \leq f(k) \quad \forall k \geq 1 \]

Let \( h(x) = f(k+1) \), \( k \in \mathbb{R} \), \( k \in \mathbb{N} \) such that

\[
h(x) > f(x) \quad \forall x > 1.
\]

By comparing the integrals, we have:

\[
\sum_{k=1}^{n-1} g(k) \leq \int_{1}^{n} g(x) \, dx \leq \sum_{k=1}^{n-1} h(k).
\]

\[
\sum_{k=1}^{n-1} g(k) = \sum_{k=1}^{n-1} f(k),
\]

\[
\sum_{k=1}^{n-1} h(k) = \int_{1}^{n} f(x) \, dx.
\]

(ii) Prove that \( \sum_{k=1}^{n} \log k \geq \int_{1}^{n} \log x \, dx \).

Apply (i) to \( f(k) = \log k \) on \( 1, n \).

\[
\sum_{k=1}^{n-1} \log k = \sum_{k=2}^{n} \log k \geq \int_{1}^{n} f(x) \, dx = \int_{1}^{n} \log x \, dx.
\]
(iii) Find the radius of convergence of \( \sum_{n=0}^{\infty} \frac{n!}{n^n} x^n \).

\[
\frac{1}{R} = \lim_{n \to \infty} \frac{(n+1)!}{(n+1)n!} \frac{n^n}{n!} = \lim_{n \to \infty} \frac{n+1}{n+1} \left(\frac{n}{n+1}\right)^n = \frac{1}{e} (1).
\]

\[
R = e
\]

(iv) Find all \( x \) where \( \sum_{n=0}^{\infty} \frac{n!}{n^n} x^n \) converges. Hint. Apply (ii) to derive a suitable inequality involving \( n!/n^n \).

\[
x = e, \quad \frac{\sum_{n=0}^{\infty} \frac{n!}{n^n} x^n}{n!} e^n \quad \text{ergo and ergo?}
\]

\[
\log a_n = \log(n!) + n - n \log n
\]

\[
= \sum_{k=1}^{\log n} \log k + \log n
\]

(by (ii)) \( \rightarrow 3 \left\{ \log x + n - n \log n \right\}
\]

\[
= (\log n - n) + 1 + n - n \log n = 1
\]

\[
\therefore a_n \geq e \quad \text{\( \forall \) } n \in \mathbb{R} \quad \text{series diverges a\( \mid \) } x = e
\]

\[
x = -e, \quad \frac{\sum_{n=0}^{\infty} \frac{n!}{n^n} (-e)^n}{n!} e^n
\]

\[
1b_n = a_n \geq e \quad \text{\( \forall \) } n \in \mathbb{N} \quad \therefore b_n \not\to 0.
\]

\[\text{series diverges at } x = -e\]

\[\therefore \text{convergence set is } (-e, e)\]