1. (10 marks) Let $R$ be the (bounded) region that lies between the curves $y = \frac{8}{x^2 + 1}$ and $y = x^2 - 1$.

a) (2 marks) Give a rough sketch of the region $R$ including the intersection points of the curves.

b) (4 marks) Express the total circumference of $R$ in terms of definite integrals. Do not simplify or evaluate these integrals.

$$l = \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \sqrt{1 + \left(16x^2(2x^3 + 1)^2ight)} \, dx + \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \sqrt{1 + 4x^2} \, dx$$

(c) (4 marks) Express the volume of the solid obtained by rotating $R$ about the line $y = -2$ as a definite integral. Do not simplify or evaluate this integral.

$$\text{Volume} = \pi \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \left[\frac{8}{(2x^3 + 1)^2} + 2\right] - (2x^3 + 1)^2 \, dx$$

For $-\frac{\sqrt{3}}{2} \leq x \leq \frac{\sqrt{3}}{2}$, $A(x)$ is area of annular region $r = x^3 + 1$, $R = \frac{8}{(2x^3 + 1)^2} + 2$.
2. (8 marks)

a) (4 marks) Write down the Simpson’s Rule approximation $S_6$ for $I = \int_1^2 e^{-6x} \, dx$.
You may express your answer as a sum involving exponential terms.

\[ \Delta x = \frac{1}{6} \quad S_6 = \frac{\Delta x}{3} \left[ f(1) + 4f\left( \frac{1}{3} \right) + 2f\left( \frac{1}{2} \right) + 4f\left( \frac{2}{3} \right) + 2f\left( \frac{5}{6} \right) + 4f(2) \right] \]

\[ = \frac{1}{6} \left[ e^{-6} + 4e^{-9/2} + 2e^{-9} + 4e^{-2} + 2e^{-12/3} + 4e^{-24} \right] \]

b) (4 marks) Without actually computing $I$, give an upper bound for $|I - S_6|$.

\[ f = e^{-6x} \quad f'(x) = (-6)^4 e^{-6x} \quad \kappa_y = \sup \{ 6^4 e^{-6x} : x \in [1, 2] \} \]

\[ = 6^4 e^{-6} \]

\[ |I - S_6| \leq \frac{\kappa_y}{180} \frac{\Delta x^4}{4} (2-1) = \frac{6^4 e^{-6}}{180} \left( \frac{1}{6} \right)^4 = \frac{e^{-6}}{180} \]
3. (10 marks)

a) (5 marks) Determine whether the improper integral \( \int_1^\infty \frac{\sin x + 1}{x^2} \, dx \) converges or diverges. Completely justify your answer.

\[
\left( \frac{\sin x + 1}{x^2} \right) \bigg|_1^\infty = \frac{2}{x^2} \bigg|_1^\infty = 2 \text{ converges}
\]

By the Comparison Test for Improper Integrals
\[
\int_1^\infty \frac{\sin x + 1}{x^2} \, dx \text{ also converges.}
\]

b) (5 marks) Determine whether the improper integral \( \int_0^\pi \frac{\sin x}{\pi - 2x} \, dx \) converges or diverges. Completely justify your answer.

\[
\left| \frac{\sin x}{\pi - 2x} \right| \to +\infty \text{ as } x \to \frac{\pi}{2}. \text{ So we break up the integral}
\]

\[
\frac{\sin x}{\pi - 2x} \int_0^{\frac{\pi}{2}} = \frac{\sin x}{\pi - 2x} \int_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \frac{\sin x}{\pi - 2x} \bigg|_0^{\frac{\pi}{2}} \text{ (by the integral of sine)}
\]

and require both these integrals to converge for convergence of the integral.

Let \( u = \frac{\pi}{2} - x \). \( \frac{\sin x}{\pi - 2x} \int_0^{\frac{\pi}{2}} = \frac{\sin x}{\pi - 2x} \int_0^{\frac{\pi}{2}} + \lim_{x \to \frac{\pi}{2}} \left( \frac{\sin \left( \frac{\pi}{2} - u \right)}{u} \right) \int_0^{\frac{\pi}{2}} \)

Now since \( u \in \left( 0, \frac{\pi}{4} \right) \), \( \frac{\sin \left( \frac{\pi}{2} - u \right)}{u} > \frac{\pi/2}{u} \); \( \int_0^{\frac{\pi}{2}} \frac{\pi/2}{u} \, du = +\infty \).

By the Comparison Test, \( \int_0^{\frac{\pi}{2}} \frac{\sin \left( \frac{\pi}{2} - u \right)}{u} \, du = +\infty \) and so \( \int_0^\pi \frac{\sin x}{\pi - 2x} \, dx \) diverges.
4. (12 marks) True or False. If True, give a proof. If False, provide, and briefly justify, a counter-example.

a) (6 marks) If \( \lim_{n \to \infty} a_n = 0 \), then the series \( \sum_{n=1}^{\infty} a_n \) is convergent.

False. \( a_n = \frac{1}{n} \). \( \lim_{n \to \infty} \frac{1}{n} = 0 \).

By the Integral Test, with \( f(n) = \frac{1}{n} \) because \( \int_{1}^{\infty} \frac{1}{x} \, dx = \lim_{a \to \infty} \ln a = +\infty \).

b) (6 marks) Let \( \{a_n\} \) be a bounded sequence, that is, for some \( R \) we have \( |a_n| \leq R \) for all \( n \). If \( \bar{a}_n = \sup \{a_k : k \geq n\} \) then \( \{\bar{a}_n\} \) is decreasing and bounded below.

True. \( \bar{a}_n \geq a_n \geq -R \) for all \( n \). \( \{\bar{a}_n\} \) is bounded below by \(-R\).

Let \( n \in \mathbb{N} \), \( \bar{a}_n \geq a_k \) for all \( k \geq n \).

\( \Rightarrow \bar{a}_n \geq a_{n+1} \) for all \( n \).

\( \Rightarrow \{\bar{a}_n\} \) is an upper bound for \( \{a_k : k \geq n\} \).

\( \Rightarrow \bar{a}_{n+1} \geq \sup \{a_k : k \geq n+1\} \).

\( \Rightarrow \bar{a}_n \geq \bar{a}_{n+1} \).

\( \Rightarrow \{\bar{a}_n\} \) is decreasing.