E.g. 2. Let \( H \) be solid hemisphere \( x^2 + y^2 + z^2 \leq 1, \ Z \geq 0 \) 
If \( H \) has density \( \rho(x,y,z) = 1 - z^2 \), find the mass, \( \text{m}(H) \), of \( H \).

Density is constant on horizontal slices \( \perp \) \( z \)-axis, so use 

slices \( \perp \) \( z \)-axis to calculate 

\[
\text{m}(H) = \int_H \rho \, dV.
\]

\( S_i = \text{horizonal slice between } z_i \text{ and } z_{i+1} \) 

where \( z_i = 2i - 1, z_{i+1} = 2i + 1 \) is a partition of \( 0, 1 \) into \( N \) equal intervals, length \( \frac{1}{N} \).

\( n^2 + 2^2 = 1 \)

\[
n = \sqrt{1 - 2^2} = \text{radius of top of } S_i
\]

\( S_i \) has volume 

\[
\Delta V_i = \pi n^2 \Delta z = \pi (1 - 2^2) \Delta z
\]

\[
\Rightarrow \text{m}(S_i) = (1 - 2^2) \pi (1 - 2^2) \Delta z \
\]

\[
\Rightarrow m(H) = \frac{1}{N} \int \pi (1 - z^2)^2 \, dz \
\]

\[
= \pi \left[ \frac{1}{3} z - \frac{1}{5} z^3 + \frac{1}{7} z^5 \right] = \frac{8 \pi}{15}
\]


Centre of Mass and Centroid

**Def.** Let \( B \subset \mathbb{R}^3 \) and \( \rho : B \to \mathbb{R} \) be the density functional of \( B \).

The \textit{x}-moment of \( B \) is \( M_x = \int_B x \rho(x, y, z) \, dV \),
the \textit{y}-moment of \( B \) is \( M_y = \int_B y \rho(x, y, z) \, dV \),
and the \textit{z}-moment of \( B \) is \( M_z = \int_B z \rho(x, y, z) \, dV \).

The mass of \( B \) is \( m(B) = \int_B \rho(x, y, z) \, dV \).

The centre of mass (c.o.m.) of \( B \) is
\[
\left( \bar{x}, \bar{y}, \bar{z} \right) = \left( \frac{M_x}{m(B)}, \frac{M_y}{m(B)}, \frac{M_z}{m(B)} \right) \in \mathbb{R}^3.
\]

Similarly for \( B \subset \mathbb{R}^d \), \( d = 3, 2 \), the c.o.m. of \( B \).

If \( d = 1 \) we call \( (\bar{x}, \bar{y}, \bar{z}) \) the centroid of \( B \).

In this case, it depends only on the geometry of \( B \).

\( (\bar{x}, \bar{y}, \bar{z}) \) is the "geographical centre" of \( B \).

\[ \text{Remark.} \quad \text{The c.o.m. is important in Mechanics.} \]

\[ \text{e.g.} \quad \text{An external Newtonian force on } B \text{ acts at } \bar{x}, \bar{y}, \bar{z}. \]

\[ \text{If } \bar{\text{c.o.m.}}(a) = \text{c.o.m. at time } t, \]

\[ \text{then } \bar{\text{c.o.m.}}(a) = \text{c.o.m. at time } t \]

\[ \bar{\text{force}} = m(B) \cdot \bar{\text{c.o.m.}}(a) = \text{m}(B) \cdot \frac{d^2 \bar{\text{c.o.m.}}(a)}{dt^2}. \]
By taking a discrete approximation to conbo's wire, one gets the same conclusion for our wire.
Q: what is centroid of \( \mathbb{R}^2 \)? Now \( f = 1 \).

A: By symmetry \( \mathbb{E} = \) .

E.g. Find centroid of hemisphere \( H \) \( x^2 + y^2 + z^2 = 1, \) \( z \geq 0 \)
by symmetry \( M_x = \int x \, dV = 0 \)

\[ \therefore \mathbb{E} = \frac{M_x}{V(H)} = 0. \]
Similarly \( \mathbb{E} = 0. \)

To find \( M_z \) let \( p_n = 2^2 \) \( 2n \) be partition of \( [0,1], \theta_{2n} = \frac{\pi}{2n} \)

\[ \Delta V_i = \text{volume of } \frac{1}{16} \text{slice} \]

\[ = \pi \theta^2 \Delta \varphi \]

\[ = \pi (1 - 2^2) \Delta z. \]

\[ r_0 = \sqrt{1 - z^2}, \]
\[ (\sqrt{1 - x^2} + z^2 - 1) \]

So \( M_z = \int z \, dV = \int \frac{1}{V(H)} z \, \Delta V_i = \frac{1}{V(H)} z \Delta V_i \times \frac{1}{2} (1 - 3x^2) \Delta z \]

\[ = \frac{1}{6} \pi (1 - 2^2) \Delta z \]

\[ = \pi \left( \frac{1}{6} - \frac{1}{4} \right) = \frac{\pi}{4} \]

\[ W(H) = \frac{1}{2} \text{Vol} (\text{ball radius 1}) = \frac{1}{2} \cdot \frac{4}{3} \pi = \frac{2}{3} \pi \]

\[ \therefore \mathbb{E} = \frac{M_z}{V(H)} = \frac{\frac{\pi}{4}}{\frac{2}{3} \pi} = \frac{3}{8} \]

\( \therefore \) c.o.m. of \( H \) is \( (0, 0, 3/8) \).
Let \( f: [a,b] \to \mathbb{R} \) be continuous.

Let \( R = \{(x,y) : a \leq x \leq b, \quad 0 \leq y \leq f(x)\} \).

Find the centroid of \( R \). Let \( x = x_0 \).

\[
\begin{align*}
\Delta M_{x_{12}} &= \text{moment of } \Delta AB_1 \text{ about } x = \int_{x_0}^{x} y \, dx \\
\Delta M_{y_{12}} &= \text{moment of } \Delta AB_2 \text{ about } y = \int_{a}^{b} x \, dy \\
\Delta A_{12} &= \text{area of } \Delta AB_1 + \Delta AB_2 \\
\Delta A_{12} &= \int_{x_0}^{x} f(t) \, dt + \int_{a}^{b} f(t) \, dt.
\end{align*}
\]

Let \( P_n = \{x_0, \ldots, x_n\} \) be a partition of \( [a, b] \), \( x_i \cdot x_{i+1} = dx \).

\[
\begin{align*}
M_x &= \int_{R} x \, dA = \frac{1}{b^2-a^2} \int_{a}^{b} x \, f(x) \, dx = \frac{1}{b^2-a^2} \Delta M_{y_{12}} \\
M_y &= \int_{R} y \, dA = \frac{1}{b^2-a^2} \int_{a}^{b} y \, f(x) \, dx = \frac{1}{b^2-a^2} \Delta M_{x_{12}}
\end{align*}
\]

Similarly, \( M_x = \sum \frac{1}{2} \Delta M_{x_{12}} \cdot dx \) and \( M_y = \sum \frac{1}{2} \Delta M_{y_{12}} \cdot dx \).

So, \( (\bar{x}, \bar{y}) = \left( \frac{M_y}{\Delta A}, \frac{M_x}{\Delta A} \right) = \left( \frac{\int_{a}^{b} y \, f(x) \, dx}{\int_{a}^{b} f(x) \, dx}, \frac{\int_{a}^{b} x \, f(x) \, dx}{\int_{a}^{b} f(x) \, dx} \right) \).
By Find centroid of \( R = \{(x,y): x^2 + y^2 \leq a^2, x, y \geq 0 \} \) - \( \frac{1}{4} \) Disk radius \( a \).

\[
\bar{y} = \frac{\int_0^a y f(x) \, dx}{\int_0^a f(x) \, dx} = \frac{\int_0^a y f(x) \, dx}{\int_0^a f(x) \, dx} = \frac{\int_0^a y \sqrt{a^2 - x^2} \, dx}{\int_0^a \sqrt{a^2 - x^2} \, dx}
\]

By symmetry \( \bar{x} = \bar{y} \).

\[
M_y = \int_0^a \frac{1}{2} (x^2 + y^2) \, dy = \int_0^a \frac{1}{2} (a^2 - x^2) \, dx = \frac{1}{2} (a^3 - \frac{a^3}{3}) = \frac{a^3}{3}
\]

\[
\bar{y} = \frac{M_y}{M_{1A}} = \frac{a^3/3}{\frac{1}{4} \frac{\pi a^2}{\frac{4}{3}}} = \frac{4}{\pi} \frac{a}{3}
\]

Centroid of \( R = \left( \frac{4}{3\pi} a, \frac{4}{\pi} a \right) \).
Pappus' Theorem (Sec. 7.6)

Pappus (c. 300 AD) (2nd Alexandrian school of mathematics)

**Pappus' Thm.**

A planar region \( R \) of area \( A \) lies on one side of a line \( L \), and is rotated about \( L \) to form a solid of revolution \( B \) with volume \( V \). (\( B \) is an irregular donut.) Let \( \bar{r} = \) distance from centroid of \( B \) to \( L \).

Then \( V = 2\pi \bar{r} A = (\text{distance travelled by centroid}) \times A \).

**Pf.**

Take \( y \)-axis = \( L \). \( R \) lies to right of \( L \).

Centroid = \((\bar{x}, \bar{y})\) = \( \left( \frac{\int x \, dA}{A}, \frac{\int y \, dA}{A} \right) \)

\( \therefore \bar{r} = \bar{y} = \) (distance from \((\bar{x}, \bar{y})\) to \( y \)-axis)

Consider a tiny box of area \( A \times \delta y \delta x \).

It sweeps out a rectangular donut of volume \((2\pi) A \delta y \delta x \) (2\pi \times \text{circumference of donut}) in \((x, y)\).

\( \therefore V = \sum_{(x, y) \in R} (2\pi) A \delta y \delta x \) is summed over a grid of points in \( R \).

\( \therefore V = \frac{\int 2\pi x \, dA}{A} \cdot A = 2\pi \bar{x} \times A \).
**Pappus' Ptnm** \( V = 2\pi \bar{r} \ \text{area of } R \)

- **Volume of solid of revolution**
  - Distance from \((x, y)\) to axis of rotation \(R\) about axis \(L\)

**By** Use Pappus' Ptnm to find volume of Torus swept out by rotating \( A = \frac{3}{4}(x, y) \cdot (x-b)^2 + y^2 = a^2 \)

about \( y \) axis, where \( b > a > 0 \).

- **Centroid** \((b, 0)\)
  - \( \bar{r} = b \)
  - \( A = 2\pi a^2 \)

So, **Volume of Torus** = \( 2\pi \bar{r} A = (2\pi b)(2\pi a^2) \)

= \( 2\pi^2 a^2 b \)

as we derived before using cylindrical shells.