Polar Curves (Sec 9.3)

Definition: Let \( r \geq 0 \) and \( \theta \in \mathbb{R} \) are polar coordinates of \((x, y)\) if
\[
(x, y) = (r \cos \theta, r \sin \theta) \quad i.e. \quad (r, \theta) \text{ is point } P \text{ with polar coordinates } (r, \theta).
\]

Remarks:
1. Geometrically, if \( P = (x, y) = (r, \theta) \), then
\[
r = \sqrt{x^2 + y^2} = \lVert \mathbf{OP} \rVert
\]
2. \( \theta \geq 0 \), then \( \theta \) is an angle in counterclockwise direction that \( \overrightarrow{OP} \) makes with pos. x-axis.
3. If \( \theta < 0 \), then \( \theta \) is an angle in clockwise direction.

\[
\begin{align*}
\text{Example:} \quad [1, -0.4] = (\cos(-0.4), \sin(-0.4)) = (\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})
\end{align*}
\]

By above, \( r \) is uniquely determined by \( P = (x, y) \), but \( \theta \) is not.

\[
\begin{align*}
\text{e.g.} \quad [1, \theta] = (r, \theta + \pi) \quad \forall r > 0 \text{ and } [0, \pi] = [0, 0] = (0, 0) \quad A \theta \text{ and } \theta + 2\pik = \theta.
\end{align*}
\]

If \( P = (x, y) \) and \( P = (r, \theta) \), then \( \theta \) is unique.

3. For sketching polar curve we allow \( r \) to be negative.

\[
\begin{align*}
\text{Example:} \quad [r, \theta] = (r \cos \theta, r \sin \theta) = \left\{ (-\cos \theta, -\sin \theta), (\cos \theta, \sin \theta) \right\}
\end{align*}
\]
\[ r = \sin 3\theta, \ \theta \geq 0 \]

Equation 1: Sketch polar graph: \( r = \sin 3\theta, \ \theta \geq 0 \)

Example 1: Sketch polar curve \( C = \{ (r, \theta) : \theta = \frac{3\pi}{4}, r > 0 \} \)

\[ r > 0 \]

Curve is an infinite line through 0 making angle \( \frac{3\pi}{4} \) with the x-axis (upper part).

Example 2: Sketch \( r = 2 \). \( r \in C = \{ (r, \theta) : \theta \in \mathbb{R} \} \)

A circle radius 2 centered at 0.

Definition 1: If \( f : [a, b] \rightarrow \mathbb{R} \) the polar graph of \( f \) is \( \{ (r, \theta) : r = f(\theta), \ \theta \in [a, b] \} \)

Equation 2: Sketch polar graph: \( r = \sin 3\theta, \ \theta > 0 \)

Example 2: Sketch polar graph: \( r = \sin 3\theta, \ \theta > 0 \)

\( \theta = \frac{\pi}{3} \)

\( \theta \in \left[ 0, \frac{\pi}{3} \right) \)

\( r \) from 0 to 1

let \( \pi \) then 1 to 0

\( f(\pi) = 0 \Rightarrow \theta = \pi/6 \) is tangent to graph at 0

\( \theta \in \left[ \frac{\pi}{3}, \frac{2\pi}{3} \right], \ r > 0, \) decreases from 0 to -1, then 1 to 0

\( \theta \in \left[ \frac{2\pi}{3}, \pi \right], \ r > 0, \) \( \theta \) from 0 to 1, then 1 to 0 then repeats.
Example 2: Sketch polar curve $C_2: r = \frac{2(1-\cos \theta)}{1-\cos^2 \theta}, \theta \geq 0$.

At $\theta = 0$, now $r \rightarrow \infty$ always.

$r \geq 0$ when $\cos \theta = -\frac{1}{2}$, i.e., $\theta = 2\pi k$ for $k \in \mathbb{Z}^+$.

Cardloid

$$f(0) = f(\pi) = 0$$

so symmetric about $x$-axis.

Example 3: Sketch polar graph $r = 2(1-\cos (\theta - \frac{\pi}{4}))$.

$C_3$ is symmetric about $x$ and $y$.

$C_3$ is rotated by $\frac{\pi}{4}$. e.g., $g(\frac{\pi}{4}) = 0 \Rightarrow \theta = \frac{3\pi}{2}$. Longest to $C_3$ at $0$. 

Trace out $C_2$. 

$C_3$ to $C_2$ rotated by $\frac{\pi}{4}$. 

$C_3$ to $C_2$, rotated by $\frac{\pi}{4}$.
**B.4** Sketch polar graph \( r = 2a \cos \theta \) \( \theta = 0 \)...

\[ r = 2a \cos \theta, \theta = 0 \] \( (r=0 \text{ is on both curves}) \)

Cartesian coordinates

\[ x^2 + y^2 = 2ax \]
\[ x^2 - 2ax + a^2 + y^2 = a^2 \]
\[ (x-a)^2 + y^2 = a^2 \] circle centered at \((a,0)\), radius \(a\)

\( f \left( \frac{\pi}{2} \right) = 0 \) so \( \theta = \frac{\pi}{2} \) tangent to \( C_4 \) at \( O \).

\[ r = f(\theta), f(\theta) = 0 \]

\[ \theta = \frac{\pi}{2} \]

\[ \lim \theta = \lim \left[ f(\theta), \theta = \frac{\pi}{2} \right] = \frac{\pi}{2} \cos c, \frac{\pi}{2} \sin c \]

\[ \lim \text{ slope } OP: \]
\[ \lim \theta = \lim \left[ \frac{f(\theta) \sin c}{\cos c} \right] \]

\[ \lim \theta = a \frac{\pi}{2} \]

So limiting angle of \( OP \) as \( \theta \to \frac{\pi}{2} \) i.e.
Area of Polar Graphs (non-rigorous)

\[ S(r, \theta_0) = \text{sector of disk of radius } r \text{ subtended by angle } \theta \text{ at } (r, \theta_0) \]

\[ A(r, \Delta \theta) = \text{area of } S(r, \theta_0) = \frac{\Delta \theta}{2\pi} \times \pi r^2 \]

\[ A(r, \Delta \theta) = \frac{\Delta \theta}{2} r^2. \]

Find area "swept out by \( r = f(\theta) \), where \( f \) is continuous.

If \( a = 0 \), \( b = 2\pi \), this is area bounded by \( r = f(\theta) \) and between the rays \( \theta = a \) and \( \theta = b \).

\[ r = f(\theta) \]

Let \( p = \Delta \psi_0, \Delta \psi_1, \ldots, \Delta \psi_n \) be a partition of \([a, b] \)

\[ \Delta A_i = \text{area swept out } \Delta \psi = f(\theta) \]

\[ \Delta \psi_i = \theta_i - \theta_{i-1} \]

\[ \Delta A_i = \frac{1}{2} f^2(\theta_i) \Delta \psi_i \]

\[ \sum_{i=1}^{N} \Delta A_i = \sum_{i=1}^{N} \frac{1}{2} f^2(\theta_i) \Delta \psi_i \]

\[ \sum_{i=1}^{N} \Delta A_i = \int_{a}^{b} \frac{1}{2} f^2(\theta) \, d\theta \]

\[ S_0 A = \int_{a}^{b} \frac{1}{2} f^2(\theta) \, d\theta \]
Eq: Warning: Area vs "Area swept out".

Let \( f(\theta) = R \) \( \theta \in [0, 2\pi] \).

1. Area swept out by \( f \):
   \[ \frac{1}{2} \int_0^{2\pi} R^2 d\theta = \frac{1}{2} \cdot 4\pi R^2 = 2\pi R^2. \]

2. Area of region enclosed by \( R = f(\theta) \):
   \[ \frac{1}{2} \int_0^{2\pi} R^2 d\theta = \pi R^2. \]

3. Area swept out by restricting \( \theta \) to \( [0, \pi/2] \):
   \[ \frac{1}{2} \int_0^{\pi/2} R^2 d\theta = \frac{1}{2} \pi R^2. \]

Eq: Find area inside 1 leaf of 3 leaf clover \( r = \sin 3\theta \).

Region is swept out for \( \theta \in [0, \pi/3] \):
\[ A = \frac{1}{2} \int_0^{\pi/3} (\sin 3\theta)^2 d\theta \]
\[ = \frac{1}{2} \left[ \frac{\pi}{3} - \frac{2}{9} \cos 3\theta \right]_0^{\pi/3} \]
\[ = \frac{1}{2} \left( \frac{\pi}{3} - \frac{2}{9} \frac{\sqrt{3}}{2} \right) \]
\[ = \frac{\pi}{12}. \]

Eq: Find area inside spiral \( r = \theta \) as shown.

Area inside \( r = 0 \), \( \theta \in [0, \pi/2] \) is swept out twice:
\[ A = \text{area swept out by } r = \theta, \theta \in [\pi/2, \pi] \]
\[ = \frac{1}{2} \int_{\pi/2}^{\pi} \theta^2 d\theta = \frac{\theta^3}{6} \bigg|_{\pi/2}^{\pi} \]
\[ = \frac{3}{16} \pi^2. \]
Example: Find area of region inside $r = a$ and $r = 2a, \theta = \frac{\pi}{3}$.

Find $0 < \theta < \frac{\pi}{2}$.

- $a = 2a, \theta = \frac{\pi}{3}$
- $\theta = \frac{\pi}{2}$, $\theta = \frac{\pi}{3}$

\[ A = 2 \times \text{area of upper } \frac{1}{3} \text{ of region} \]
\[ = \pi \int_{0}^{\pi/3} \left( \frac{1}{2} a^2 d\theta + \frac{1}{2} (2a \cos \theta)^2 d\theta \right) \]
\[ = \frac{\pi}{3} a^2 + 4a^3 \int_{0}^{\pi/3} \frac{1 + \cos 2\theta}{2} d\theta \]

\[ (B1P2) = a^2 \left[ \frac{2\pi}{3} - \frac{\sqrt{3}}{2} \right] \]