Thm 9.26. Consider a power series \( \sum_{n=0}^{\infty} a_n (x-c)^n \).

\[ \exists R \in [0, \infty) \text{ s.t. (i) } |x-c| < R \Rightarrow \sum_{n=0}^{\infty} a_n (x-c)^n \text{ converges absolutely} \]

\[ (ii) \ |x-c| > R \Rightarrow \sum_{n=0}^{\infty} a_n (x-c)^n \text{ diverges.} \]

Q: How can we find \( R \)? Proof didn’t help here.
A: Apply Root or Ratio Test.
How can we find $R$? Convention: \( \frac{1}{0} = 0, \frac{1}{\infty} = 0. \)
\[
\infty \cdot \frac{1}{\infty} = \infty \quad \text{if} \ c > 0.
\]

**Theorem 9.27** Let $R$ be the radius of convergence of
\[
\sum_{n=0}^{\infty} a_n (x-c)^n.
\]

(a) \( |a_n|^{1/n} \to \infty \) \( \Rightarrow \) \( R = 0 \) \( \infty \in (0, \infty) \).

(b) \( |a_n| \to \infty \in (0, \infty) \) \( \Rightarrow \) \( R = \frac{1}{c} \in (0, \infty) \).

**Proof.** (a) \( \Rightarrow \) (b) by Lemma 9.17 (hypothesis \( \Rightarrow \) hypothesis \( \Rightarrow \).

(a) Let \( x \neq c \). Apply the Root Test to \( \sum_{n=0}^{\infty} |a_n (x-c)^n| \).

\[
|a_n (x-c)^n|^{1/n} = |a_n|^{1/n} |x-c| \to 0 |x-c| = p \in (0, \infty).
\]

If \( p = \infty \), \( |x-c| = \infty \) by \( x \neq c \) and our convention.

(b) \( p < 1 \Rightarrow \) \( |x-c| < 1 \Rightarrow |x-c| < 1/\infty \) and

14. \( p = 1 \Rightarrow \) \( |x-c| = 1 \Rightarrow |x-c| = 1/\infty \)

The second equivalence holds for \( p < 1 \) and \( p = \infty \) since

either both sides hold trivially or fail trivially. Hence \( x \neq c \) is used in the \( p = \infty \) cases.

Now use (14, 15) and the Root Test to conclude

(14) \( |x-c| < 1/\infty \Rightarrow \sum_{n=0}^{\infty} a_n (x-c)^n \) converges absolutely

(15) \( |x-c| > 1/\infty \Rightarrow \sum_{n=0}^{\infty} a_n (x-c)^n \) diverges.

Thus \( R = 1/\infty = 0. \)
158.) Find the radius of convergence, \( R \), centre of convergence, \( c \), and set of convergence, \( C \) for

\[
\sum_{n=1}^{\infty} \frac{n^2 (2x - 3)^n}{a_n} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{2^n n^2 (x - 3)^n}{a_n}
\]

\[
c = \frac{3}{2}, \quad \left| \frac{a_{n+1}}{a_n} \right| = \frac{\sum_{n=1}^{\infty} \frac{n^2}{a_n}}{\sum_{n=1}^{\infty} \frac{2n^2}{a_n}} = 2 \left( \frac{\frac{1}{n+1}}{\frac{1}{n}} \right)^2 \to 2.
\]

By 9.27(6) \( R = \frac{1}{2} \).

Absolute convergence holds if \( |x - \frac{3}{2}| < \frac{1}{2} \iff x \in (\frac{3}{2} - \frac{1}{2}, \frac{3}{2} + \frac{1}{2}) = (0, 2) \)

Divergence holds if \( |x - \frac{3}{2}| > \frac{1}{2} \iff x \not\in [0, 2] \)

\( x = 1 \)

\[
\sum_{n=1}^{\infty} \frac{(2 \cdot 1 - 3)^n n^2}{a_n} = \sum_{n=1}^{\infty} (-1)^n n^2 \quad \text{which is absolutely convergent (p-series)}
\]

\( x = 2 \)

\[
\sum_{n=1}^{\infty} \frac{(2 \cdot 2 - 3)^n n^2}{a_n} = \sum_{n=1}^{\infty} (-1)^n n^2 \quad \text{which is absolutely convergent.}
\]

\( \therefore C = [0, 2] \) and absolute convergence holds on \( \mathbb{R} \setminus \{0, 2\} \).
Find the interval of convergence of \( \sum_{n=1}^{\infty} \frac{n^2}{n^3} x^n \).

\( a_n = n^2, \quad a_{n+1} = 0 \) so \( a_{n+1}/a_n \) won't converge.

\( a_{2n} = n, \quad a_{2n-1} = 0 \) so Lemma 9.22 implies \( a_{n+1}/a_n \) converges.

**Solution:** Let \( y = x^2 \). Series becomes \( \sum_{n=1}^{\infty} \frac{n^2}{n^3} y^n \), i.e. \( a_n = x^2 \)

\( a_n \left( \frac{n}{n^2} \right)^2 \xrightarrow{n \to \infty} 1 = 1 \). \( \frac{d}{dx} \)

\[ e^{y} = e^{x^2} \] converges for \( |x| \leq 1 \)

\[ \sum_{n=1}^{\infty} \frac{n^2}{n^3} x^n \] converges for \( |x| < 1 \).

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**Thm. 9.24. (Differentiation + Integration of Power Series; DIP)**

Assume \( f(x) = \sum_{n=0}^{\infty} a_n x^n \) converges for \( |x| < r \) where \( r > 0 \).

(a) \( f'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} \) for \( |x| < r \).

(b) \( \int f(x) \, dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} \) for \( |x| < r \).

We may differentiate and integrate power series like polynomials.

---

**Corollary 9.29.** If \( f(x) = \sum_{n=0}^{\infty} a_n x^n \) converges for \( |x| < r \), then \( f \) is infinitely differentiable on \( (0, r) \),

and \( \frac{d^n}{dx^n} f(x) = n! \cdot a_n \) for \( |x| < r \).

---

**Proof:** \( f'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} \), \( f(0) = 0 \) by 9.23(a).

Apply 9.23(b) to this power series: \( f''(x) = \sum_{n=2}^{\infty} \frac{n(n-1)}{n!} a_n x^{n-2} \).

In general, induct on, \( n \).
Thm 9.29. Assume $r > 0$ and $f(z) = \sum a_n z^n$ for $|z| < r$.(cvb).
(a) $f'(z) = \sum n a_n z^{n-1}$ for $|z| < r$.
(b) If $g(x) = \int_0^x f(t) dt = \sum a_n x^n$, for $|x| < r$.

Write $R(f)$ for the radius of convergence of the above power series for $f$. (We should write $R(\log f)$!)
Similarly for $R(f')$ and $R(1/f)$.

Cor 9.19 Assume $f(z) = \sum a_n z^n$ has $R(f) > 0$.

Thus $R(f') = R(1/f) = R(f)$.

Pl: we may take $r = R(f)$ in $|z| < r$.

By (a) the power series for $f'$ converges for $|z| < R(f)$ and so $(1) R(f') \geq R(f)$.

Similar reasoning gives $(2) R(1/f) \geq R(f)$.

Apply 9.28(a) to the power series rule with $r = R(f)$ gives,

If $f' = f$ (i.e. have the same power series)

(weakly FTC as its divisible and so its by $a$)

$R(f') = R(f)$ (same power series)

$\geq R(1/f)$ by $(1)$ with $f$ in place of $f$.

$\geq R(f)$ by $(2)$

$\Rightarrow (3) R(f) = R(1/f)$. 

(4) $I_f(a) = f(a) - a_0$ (i.e. the power series obtained by integrating the power series $f(x)$ is $\frac{1}{m} a_0 z^m$ ; for functions this is FTC.)

So $R(f') = R(1/f)$ by $(3)$ but $f(0) = a_0$,

$= R(f - a_0)$ by $(4)$

$= R(f)$ (trivial) $\square$
Ex. 11 Find a power series representation for \( \frac{1}{(1-x)^2} \).

\[
\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \forall \, 1 \leq 1
\]

\[
\frac{1}{(1-x)^2} = \frac{d}{dx} \frac{1}{1-x} = \sum_{n=1}^{\infty} n \cdot x^{n-1} \quad \forall \, 1 \leq 1 \text{ by 9.29(b)}
\]

\[
\left( = \sum_{m=0}^{\infty} (m+1)x^m \quad m=n \right) \text{ only if you need convincing it's a power series}
\]

Ex. 2 Evaluate: \( \sum_{n=1}^{\infty} nx^n \), \( \forall \, 1 \leq 1 \).

\[
\sum_{n=1}^{\infty} nx^n = n \sum_{n=1}^{\infty} x^n = x \sum_{n=1}^{\infty} nx^{n-1} \quad \text{(algebra)}
\]

\[
= \frac{x}{(1-x)^2} \quad \text{by above. (PG)}
\]

\[
\sum_{n=1}^{\infty} \frac{n^2 x^{n-1}}{n} = \frac{d}{dx} \frac{x}{(1-x)^2} = \frac{1+x}{(1-x)^3}
\]

\[
\sum_{n=1}^{\infty} \frac{n^2 x^n}{n} = x \sum_{n=1}^{\infty} nx^{n-1} = \frac{x+x^2}{(1-2)^3}
\]

0. \( \frac{1}{2} \sum_{n=1}^{\infty} \frac{n^2 x^n}{n} = \frac{1}{2} \cdot \frac{x+x^2}{(1-2)^3} = 6 \).

Ex. 3 Find a power series representation for \( \log(1+x) \).

\[
\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{for} \, 1 \leq 1
\]

Let \( x = -t \): \( \frac{1}{1+t} = \sum_{n=0}^{\infty} (-1)^n t^n \quad \text{for} \, 1 \leq 1 \).

By (9.29(b)) if \( 1 \leq 1 \) then \( \left( \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \right) = \sum_{m=1}^{\infty} \frac{(-1)^{m-1} x^m}{m} \).

Know: \( \log(1+x) = \sum_{m=1}^{\infty} \frac{(-1)^{m-1} x^m}{m} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \quad \forall \, 1 \leq 1 \)