Lecture 35 (Section 9.4) March 16

Def. \( \sum_{n=0}^{\infty} a_n \) converges absolutely iff \( \sum_{n=0}^{\infty} |a_n| \) converges.

Thm 9.20 \( \sum_{n=0}^{\infty} a_n \) absolutely convergent \( \Rightarrow \) \( \sum_{n=0}^{\infty} a_n \) is convergent.

Ex. Find all \( x \in \mathbb{R} \) s.t. \( \sum_{n=0}^{\infty} \frac{x^n}{n!} \) is convergent.

We showed on Fri. that \( x \in \mathbb{R} \), \( \sum_{n=0}^{\infty} \frac{x^n}{n!} \) converges.

So \( \sum_{n=0}^{\infty} \frac{x^n}{n!} \) is absolutely convergent, hence convergent for all \( x \in \mathbb{R} \).

Question: Does the converse to Thm 9.20 hold?

The following shows the converse fails.

Thm 9.21 (Alternating Series Test -- AST)
Let \( b_n \downarrow 0 \). Then \( \sum_{n=1}^{\infty} (-1)^{n-1} b_n \) converges.

Ex. Let \( p > 0 \), \( b_n = \frac{1}{n^p} \downarrow 0 \).
By AST \( \sum_{n=1}^{\infty} (-1)^{n-1} b_n \) converges.

If \( p = 1 \), \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \) is convergent.

If \( p \leq 1 \), \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p} \) is convergent but not absolutely convergent.
Def. A series $\sum_{n=0}^\infty a_n$ is conditionally convergent if it is convergent but not absolutely convergent.

To prove AST we will need an elementary result whose proof is left as an E.F.S.

**Lemma 9.22** \[ \forall A \in [0,\infty) \, \lim_{n \to \infty} b_n = l \iff b_{2n} \to l \text{ and } b_{2n-1} \to l. \]

**Proof of AST.** Let $S_n = \sum_{k=1}^n (-1)^{k-1} b_k := \sum_{\mu=1}^n a_\mu$.

**Claim:** 1. $\{S_{2n}\}$ is 18 1.

\[
S_{2(n+1)} - S_{2n} = a_{2n+1} + a_{2n+2} = (1-2n) b_{2n+2} + (-1)^{2n+1} b_{2n+1} = b_{2n+1} - b_{2n+2} \geq 0 \iff b_n \leq 0.
\]

$\therefore \{S_{2n}\}$ is a 18 1.

**Claim:** 2. $\{S_{2n-1}\}$ is 18 1.

\[
S_{2n} - S_{2n-1} = q_{2n} + a_{2n+1} = (1-2n) b_{2n+1} + (-1)^{2n} b_{2n} \geq 0.
\]

$\therefore \{S_{2n-1}\}$ is 18 1.

We also have 3. $S_{2n-1} - S_{2n} = q_{2n} = (-2n b_{2n} > 0 \quad \text{i.e. } S_{2n-1} > S_{2n}$

So 0, 2, 3 imply 4. $S_1 > S_2 > \cdots > S_{2n-1} > S_{2n} > S_{2(n+1)} = \cdots > S_4 \geq S_2$.

So $S_{2n} \geq l$ and bounded above by $S_1$; $\{S_{2n-1}\}$ is 18 and bounded below by $S_2$. 


By the Monotone Sequence Thm 9.4 \( s_{2n} \to s_{	ext{even}} \in \mathbb{R} \)
\[ s_{	ext{even}} - s_{	ext{odd}} = \lim_{n \to \infty} s_{2n} - \lim_{n \to \infty} s_{2n-1} \]
\[ = \lim_{n \to \infty} s_{2n} - s_{2n-1} \quad \text{(Algebra of Limits)} \]
\[ = \lim_{n \to \infty} (n+1)^{2n-1} b_{2n} = 0 \]
\[ \therefore s_{	ext{even}} = s_{	ext{odd}} = s. \]

By Lemma 9.22 \( s_n \to s \in \mathbb{R}, \) \( 1 \leq n \leq \sum_{n=1}^{\infty} b_n = S \). \( \blacksquare \)

**Remark 9.23. (Alternating Series Bounds)** Let \( b_n \geq 0 \) and \( \sum_{n=1}^{\infty} b_n \)
be as above \( (b_n : b_n = \sum_{n=1}^{\infty} b_n) \). By \( 1, 2 \) \( s_{2n+1} \leq s \leq s_{2n} \leq s_{2n-1} \leq s. \)

So for all \( n \in \mathbb{N} \)
\[ s_{2n} \leq s \leq s_{2n+1} \leq s_{2n-1} \]

This implies \( (1) \) \[ |s_{2n} - s| = s - s_{2n} \leq s_{2n+1} - s_{2n} = b_{2n+1} \]
and \( (2) \) \[ |s_{2n-1} - s| = s_{2n-1} - s \leq s_{2n} - s_{2n-1} = (1)^{2n-1} b_{2n} = b_{2n} \]

\[ (1) \Rightarrow |s_m - s| \leq b_{m+1} \quad \forall m \in \mathbb{N} \] (Alternating Series bounds)

[Can you see this from Figure?]

\( (A\text{.9}3) \)
9.23: \( |S_n - s| \leq \frac{1}{n+1} b_{n+1}, \quad s = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} b_k}{n^2} \) (convergent by AST)

Eq. 1 \( S_n = \sum_{k=1}^{n} \frac{(-1)^{k-1} b_k}{n^2}, \quad S = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} b_k}{n^2} \)

Find \( N \) (as small as you can) s.t. \( |S - S_N| \leq \frac{1}{100} \)

by (AST) \( |S - S_N| \leq \frac{1}{8(n+1)^2} \)

So it suffices that \( \frac{1}{(n+1)^2} \leq \frac{1}{100} \iff (n+1)^2 \geq 10^2 \iff N = 9 \)

Eq. 2.1 Find all \( x \) s.t. \( \sum_{n=1}^{\infty} \frac{x^n}{n} \) is convergent / \( x \) is absolutely convergent.

Fix \( x > 0 \) (well-defined) and let \( a_n = \frac{2^n}{2^n n} \).

Use the Ratio Test to test for absolute convergence

\[
\frac{|a_{n+1}|}{|a_n|} = \frac{\frac{2^{n+1}}{2^{n+1} (n+1)}}{\frac{2^n}{2^n n}} = \frac{2}{n+1} \to \frac{2}{2} = 1 \Rightarrow p = \frac{2}{2} = 1
\]

so \( |x| < 2 \iff p < 1 \Rightarrow \sum_{n=1}^{\infty} \frac{x^n}{n} \) converges absolutely

\( |x| > 2 \iff p > 1 \Rightarrow \lim_{n \to \infty} \left| \frac{x^n}{n} \right| = \infty = \sum_{n=1}^{\infty} \frac{x^n}{n} \) diverges.

\( x = 2 \); \( \sum_{n=1}^{\infty} \frac{2^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty \)

\( x = -2 \); \( \sum_{n=1}^{\infty} \frac{-2^n}{n} = \sum_{n=1}^{\infty} \frac{(-2)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \) converges by AST

But \( \sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{2^n n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} = \infty = \sum_{n=1}^{\infty} \frac{x^n}{n} \) converges conditionally

Conclusion: \( \sum_{n=1}^{\infty} \frac{x^n}{2^n n} \) converges iff \( x \in (-2, 2) \).

It converges absolutely iff \( x \in (-2, 2) \).

And so it converges conditionally iff \( x = -2 \).
Recall: \( x^+ = \max(x, 0), \ x^- = \max(-x, 0) \), \( x = x^+ - x^- \), \( |x| = x^+ + x^- \).

Prop. 9.24. Let \( \sum a_n \) be convergent.
Then \( \sum a_n \) is conditionally convergent iff \( \sum a_n^+ = \sum a_n^- = \infty \).

(So there is a lot of cancellation to make \( \sum a_n \) converge.)

E.g. \( a_n = (-1)^{n-1} \frac{1}{n} \) alternating harmonic series.
\[ a_n^+ = \begin{cases} \frac{1}{n} & \text{odd} \\ 0 & \text{even} \end{cases} \]
\[ a_n^- = \begin{cases} 0 & \text{even} \\ \frac{1}{n} & \text{odd} \end{cases} \]

\[ s_0 \sum a_n^+ = \lim_{N \to \infty} \sum_{n=1}^{2N} \frac{1}{n} = \lim_{N \to \infty} \sum_{k=1}^{N} \frac{1}{2k-1} = \frac{\infty}{2} = \infty \]

Similarly \( s_0 \sum a_n^- = \lim_{N \to \infty} \sum_{n=1}^{2N} 0 = \infty \left( \text{e.g. use } \frac{1}{2k-1} \geq \frac{1}{2k} \right) \)

Proof \((\Rightarrow)\) Assume \( \sum a_n \) convergent and \( \sum |a_n| = \infty \).
Proceed by contradiction and assume \( \sum a_n^+ \) is convergent.
\( a_n = a_n^+ - a_n^- \) implies \( a_n^- = a_n^+ - a_n \). So \( \sum a_n^- = \sum a_n^+ - \sum a_n = \sum a_n^+ - \sum a_n^- \) is also convergent by Algebra of Series (L Th 9.9).
\[ \sum |a_n| = \sum a_n^+ + \sum a_n^- = \sum a_n^+ + \sum a_n^- \text{ is also convergent (L Th 9.9)} \]
This is a contradiction so \( \sum a_n^+ \) must be divergent.

Now replace \( a_n \) with \( -a_n \) in the above argument (the hypotheses still hold, right?),
\[ s_0 \sum (-a_n)^+ = \infty \text{ but } (-a_n)^+ = a_n^- \text{ (check definition)} \]
\[ \sum a_n^- = \infty. \]
1. Assume \( \sum a_n^+ = \sum a_n^- = \infty \) (and \( \sum a_n \) convergent).

\[ |a_n| \geq a_n^+ \text{ so by the Comparison Test (Th 9.15)} \]

\[ \sum |a_n| = \infty . \]

\[ \Rightarrow \sum a_n \text{ is conditionally convergent.} \]

**Def:** \( \{a_{n_j}\} \) is a rearrangement of \( \{a_n\} \) if \( j \rightarrow n_j \) is a bijection from \( \mathbb{N} \) to \( \mathbb{N} \).

Q: Assume \( \sum a_{n_j} \) is the sum of a rearrangement of \( \{a_n\} \)

where \( \sum a_n \) is c. v. t. \( \Rightarrow \sum a_{n_j} = \sum a_n ? \)

b. Is c. v. t. addition commutative?

A. No in general.

\( \sum a_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots \). \( a_n = (-1)^{n+1} \frac{1}{n} \).

By ABT \( \sum \frac{1}{2n-1} - \frac{1}{2n} \) is c. v. t. and \( s_2 = s = s_3 \) \( \frac{1}{6} \).

\[ \frac{1}{4} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots \]

\( = 1 - \frac{3}{2} + \frac{1}{2} - \frac{1}{2} + \frac{1}{5} - \frac{1}{6} + \frac{1}{2} - \frac{1}{2} + \cdots \) (odd c. v. t., odd c. v. t. -)

\( = (1 - \frac{3}{2}) + (\frac{1}{2} - \frac{1}{2}) + (\frac{1}{5} - \frac{1}{6} + \frac{1}{2} - \frac{1}{2}) - \frac{1}{2} - \cdots \)

\( = \frac{1}{2} - \frac{1}{2} + \frac{1}{2} - \frac{1}{2} + \frac{1}{2} - \frac{1}{2} + \cdots \)

\( = \frac{5}{2} - \frac{3}{2} + \frac{1}{2} - \frac{1}{2} + \cdots \)

\( \Rightarrow s = \frac{3}{2} \Rightarrow s \neq 0 \text{ contradicting } s = \frac{1}{2}, \text{ so (b) fails.} \)
This is possible since \( \frac{2}{n} \) converges conditionally.

**Thm 9.26.** Let \( \sum a_n \) be cvg.

If \( \sum a_n \) converges absolutely, then a rearrangement \( \sum a_{k_j} \) of \( a_n \), \( \sum a_{k_j} = \frac{2}{n} \).

II Assume \( \sum a_n \) is conditionally cvg. If \( \varepsilon(\theta) < 1 \)
a rearrangement \( \sum a_{k_j} \) of \( a_n \) s.t. \( \frac{2}{n} a_{k_j} = \varepsilon \).

**Pr.** Sec. Spivak Ch 22 Th 9.7 [1st edition]. Prop 9.246 key idea
To illustrate idea int6) consider \( a_n = (-1)^{n-1} \frac{1}{n} \), \( \varepsilon = 42 \).

To find \( \sum a_{k_j} \) add enough positive (odd) terms s.t. 
\[
\frac{1}{1} + \frac{1}{3} + \cdots + \frac{1}{2n-1} = \frac{\pi}{2} \frac{1}{n} > 42. \quad \text{(In general use Prop 9.24)}
\]

Then add enough (one will do in this case) negative terms s.t.
\[
\frac{1}{2} \frac{1}{2n} - \frac{1}{2} < 42. \quad \text{(Prop 9.24 in general)}
\]

Now add enough odd terms so that \( \frac{N^2}{\varepsilon} a_{k_j} > 42 \).
Continue. Since \( a_n \to 0 \), \( \sum a_{k_j} a_{k_j} = 42 \).