Math 121 Assignment 9 Due Wed. March 27 at start of class

You may use any results stated in class (except when asked to prove such a result!).

1. Determine the centre, interval and radius of convergence of the following power series:
   (a) \( \sum_{n=0}^{\infty} \frac{e^n}{n^3}(1 - 2x)^n \).
   (b) \( \sum_{n=0}^{\infty} \frac{x^{3n}}{\sqrt{n+1}}. \) Hint. Try setting \( y = x^3 \).

Solutions. (a) The above series is \( \sum_{n=1}^{\infty} \frac{(-2e)^n}{n^3}(x - \frac{1}{2})^n \equiv \sum_{n=1}^{\infty} a_n(x - \frac{1}{2})^n \).

We have \( \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \to \infty} 2e \left( \frac{n+1}{n+2} \right)^3 = 2e \). So the radius of convergence is \((2e)^{-1}\) and the centre is \( \frac{1}{2} \). At \( x = \frac{1}{2} \pm (2e)^{-1} \) the series of absolute values becomes \( \sum_{n=1}^{\infty} n^{-3} \) which is summable. Hence we have absolute convergence at the two endpoints and the interval of convergence is \([\frac{1}{2} - (2e)^{-1}, \frac{1}{2} + (2e)^{-1}]\).

(b) Note that the coefficients which are not multiples of 3 are 0 and so we cannot apply the ratio or root tests directly to find the radius of convergence. If we let \( y = x^3 \), the series becomes \( \sum_{n=0}^{\infty} (n+1)^{-1/2} y^n \). Now analyzing this power series we see \( \frac{1}{R} = \lim_{n \to \infty} \frac{n^{1/2}}{(n+1)^{1/2}} = 1 \) and so this series has radius of convergence 1. It converges at \( y = -1 \) by the alternating series test and diverges at \( y = 1 \) by the integral test. So the latter series converges iff \( y \in [-1, 1) \) and so the original series converges iff \( x^3 \in [-1, 1) \) iff \( x \in [-1, 1) \). This is the interval of convergence. The radius of convergence is 1 and the centre is 0.

2. Find the sum of the given series or show that the series diverges:
   (a) \( S_1 = \sum_{n=1}^{\infty} \frac{1}{n^2} \).
   (b) \( S_2 = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \cdot (-1)^n \).

Solutions. (a) Integrating the geometric sum \( \frac{1}{1-t} = \sum_{n=0}^{\infty} t^n \) \((|t| < 1)\), gives for \( |x| < 1 \), \( \int_0^x \frac{1}{1-t} dt = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{x^n}{n} \). The integral on the left is \( -\log(1-x) \). So set \( x = 1/2 \) to get \( S_1 = \log 2 \). (Alternatively you could use the series expansion for \( \log(1 + x) \) derived in class.)

(b) Recall that \( \sin x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \cdot (-1)^n \) for all \( x \), so setting \( x = \pi/4 \) we see that \( S_2 = \sin(\pi/4) = 1/\sqrt{2} \).

3. Find power series representations for the following functions. On which interval is your representation valid? Justify all answers. It is suggested that you start with \( \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \) and reason in the appropriate manner.
4. Find Taylor series for the following functions and find the intervals on which
the series representation is valid (justify your answers). Recall that any valid
power series representation at a point must be the Taylor series at the point.

(a) $\frac{x^3}{1-2x^2}$ in powers of $x$.
(b) $\log x$ in powers of $x - 4$. (Take a look at Example 4 on p. 547.)

**Solution.** (a) We know that $1/(1 - y) = \sum_{n=0}^{\infty} y^n$ iff $|y| < 1$. So setting $y = 2x^2$ we see that $\frac{x^3}{1-2x^2} = x^3 \sum_{n=0}^{\infty} (2x^2)^n = \sum_{n=0}^{\infty} 2^n x^{3+2n}$ iff $|2x^2| < 1$ iff $|x| < 1/\sqrt{2}$.

(b) Recall that $\log(1 + y) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{y^n}{n}$ iff $y \in (-1, 1)$. So setting $y = \frac{x-4}{4}$ we see that for $(*)$ $(x - 4)/4 \in (-1, 1)$, and only for these values, we have:

\[
\log(x) = \log((4 + (x - 4)/4)) = \log 4 + \log\left(1 + \frac{x - 4}{4}\right)
\]
\[
= \log 4 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(\frac{x - 4}{4}\right)^n
\]
\[
= \log 4 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n4^n} (x - 4)^n.
\]

Simplifying (*), we see this representation is valid iff $x \in (0, 8]$.

4. Find Taylor series for the following functions and find the intervals on which
the series representation is valid (justify your answers). Recall that any valid
power series representation at a point must be the Taylor series at the point.

(a) $\arctan(5x^2)$ about $x = 0$.
(b) $\frac{1+x^3}{1+x^4}$ about $x = 0$.
(c) $\cos^2 x$ about $x = \pi/8$.

**Solution.** (a) Recall that $\arctan y = \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n+1}}{2n+1}$ iff $y \in [-1, 1]$. Therefore for $5x^2 \in [-1, 1]$, and only for these values, we have $\arctan(5x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(5x^2)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{5^{2n+1}x^{4n+2}}{2n+1}$. The above condition on $x$ simplifies to $x \in [-1/\sqrt{5}, 1/\sqrt{5}]$. By uniqueness of the powers series representation, the above power series must be the Taylor series for $\arctan(5x^2)$.

(b) For $|x| < 1$, and only for these values we have

\[
\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n.
\]
Therefore we have

\[
\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n (x^{2n} + x^{2n+3}) = \sum_{n=0}^{\infty} (-1)^n x^{2n} + \sum_{k=2}^{\infty} (-1)^k x^{2k-1}.
\]
In the last we have set $k = n + 2$. The series representation is valid iff $x \in (-1, 1)$.

(c) For all $x$ we have $\cos 2x = \cos(2(x - \pi/8) + \pi/4) = \cos(2(x - \pi/8)) \cos(\pi/4) - \sin(2(x - \pi/8)) \sin(\pi/4) = \frac{1}{\sqrt{2}}[\cos 2y - \sin 2y]$, where $y = x - \pi/8$. Therefore
we have for all $x$,

\[
\cos^2 x = \frac{1 + \cos(2x)}{2}
\]

\[
= \frac{1}{2} + \frac{1}{2\sqrt{2}} [\cos 2y - \sin 2y]
\]

\[
= \frac{1}{2} + \frac{1}{2\sqrt{2}} \left[ \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} y^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 2^{2n+1} y^{2n+1}}{(2n+1)!} \right]
\]

\[
= \frac{1}{2} + \frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} \left[ \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n} (x - \pi/8)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 2^{2n+1} (x - \pi/8)^{2n+1}}{(2n+1)!} \right].
\]

5. Find the MacLaurin series for $I(x) = \int_0^x \frac{\sin t}{t} \, dt$ and use it to find $I(1)$ correct to 3 decimal places.

**Solution.** Note that although $\sin t/t$ is not defined at 0, it approaches 1 as $t \to 0$ and is uniformly bounded on $(0, \infty)$. So as in Q 6 on HW6 the function when defined to be 1 (or 42) at $t = 0$ is integrable. We will define it to be 1 at $t = 0$. So there is no need to invoke improper integration. So $\sin t/t = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!}$ for all $t \in \mathbb{R}$ (At $t = 0$ both sides are 1.). Therefore

$I(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$. If $b_n = \frac{1}{(2n+1)(2n+1)!}$, and $s_n = \sum_{k=0}^{n} b_k$, then by alternating series bounds (note that $b_n$ decreases to 0) we have $|I(1) - s_N| \leq b_{N+1}$ and so we seek the smallest $N$ so that $b_{N+1} < .0005$. That is so that $(2N + 3)(2N + 3)! > 2000$. $7 \times 7! = 35,280 > 2000$ and so we may take $N = 2$. Therefore $I(1) \approx 1 - \frac{1}{3!} + \frac{1}{5!} = \frac{1703}{1800} = .94611$ with an error of at most $1/35,280 < .00003$. 

6. Consider $f(x) = \sum_{n=0}^{\infty} x^{2n}$ for $x$ in the interval of convergence of this power series.

(a) Find the interval of convergence of the power series.

(b) Find $f^{(100)}(0)$.

(c) Find $f^{(16)}(0)$.

(d) Find $\lim_{x \to 0} \frac{f(x) - x(1+x)}{x^4}$.

**Solution.** (a) If $|x| \geq 1$, $\lim_{n \to \infty} |x|^{2n}$ is not 0 and so the series must diverge. On the other hand an easy induction shows that $2^n \geq n$ for all non-negative integers $n$. Therefore for $|x| < 1$, $|x|^{2n} \leq |x|^n$ and so by the comparison test $\sum_{n=0}^{\infty} |x|^{2n} \leq \sum_{n=0}^{\infty} |x|^n < \infty$. This shows the interval of convergence is $(-1, 1)$.

(b) The above series must be the MacLaurin series for $f$. Since the $x^{100}$ term in the power series is 0, it follows that $f^{(100)}(0)/100! = 0$ and so the hundredth derivative of $f$ at 0 is 0.
(c) As the $x^{16}$ coefficient in the Taylor series of $f$ is 1, we know that $f^{(16)}(0)/16! = 1$ and so $f^{(16)}(0) = 16!$.

(d) Note that $f(x) = x + x^2 + \sum_{n=2}^{\infty} x^{2n} = x(1 + x) + x^4 g(x)$, where $g(x) = \sum_{n=2}^{\infty} x^{2n-4}$ for $|x| < 1$. The latter series is convergent as in (a) for such $x$, and so $g$ is infinitely differentiable and in particular is continuous with $g(0) = 1$. Therefore

\[
\lim_{x \to 0} \frac{f(x) - x(1 + x)}{x^4} = \lim_{x \to 0} \frac{x^4 g(x)}{x^4} = \lim_{x \to 0} g(x) = 1,
\]

where in the last line we use the fact that $g$ is continuous at 0.

7. Practice: Ex 9.6 #17, 20, 25, 37, 43