Math 121 Assignment 8

This assignment is due in class on Fri. March 14 but note that this material may be on the midterm on March 12. It is possible that only a selection of the questions will be graded.

1. In class we considered the triangle $R$ with vertices $(0,0)$, $(1,0)$ and $(0,1)$ and the solid $S$ obtained by rotating it about the line $y = 2$. We used the method of cylindrical shells to show the volume of $S$ is $5\pi/3$. Calculate the same volume using slices perpendicular to the $y$-axis. CORRECTION: AS NOTED IN CLASS THIS SHOULD BE THE LINE $x = 2$—SOME MAY HAVE CORRECTED IT BY TAKING SLICES PERP. TO THE X-AXIS, WHICH IS FINE TOO. AS A RESULT THIS QUESTION WAS NOT GRADED.

2. Find the arc length of $y = x^2/2$ from $x = 0$ to $x = 2$.

$$\ell = \int_0^2 \sqrt{1 + x^2} \, dx = \int_0^{\tan^{-1} 2} \sec^2 \theta \, d\theta,$$

where $x = \tan \theta$. Since $\sec(\tan^{-1} 2) = \sqrt{5}$, we get

$$\ell = \frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \log |\sec \theta + \tan \theta|_0^{\tan^{-1} 2} = \sqrt{5} + \frac{1}{2} \log(\sqrt{5} + 2).$$

3. Find the length of the “closed” curve $x^{2/3} + y^{2/3} = a^{2/3}$.

By symmetry the arc length will be 4 times the length of the curve in the first quadrant. There we have $y = (a^{2/3} - x^{2/3})^{3/2}$ for $0 \leq x \leq a$. Therefore $(y')^2 = (a^{2/3} - x^{2/3})^{-2/3} = (a/x)^{2/3} - 1$, and so

$$\ell = 4 \int_0^a \sqrt{(a/x)^{2/3}} \, dx = 4a^{1/3} \int_0^a x^{-1/3} \, dx = 4a^{1/3} \frac{3}{2} a^{2/3} = 6a.$$

4. Find the limits of the following sequence (finite, $+\infty$ or $-\infty$). Justify your answers.

(a) $\left\{2 + \frac{(-1)^n}{n}\right\}$.

$$\lim_{n \to \infty} 2 + \frac{(-1)^n}{n} = 2 + \lim_{n \to \infty} \frac{(-1)^n}{n} = 2.$$ In the last equality we could note that $|(-1)^n/n| = 1/n \to 0$ as $n \to \infty$, from which it follows by the definition of limit that $\lim_{n \to \infty} \frac{(-1)^n}{n} = 0$. Or you could use the Squeeze theorem with $-1/n \leq (-1)^n/n \leq 1/n$.

(b) $\left\{\frac{1-n^2}{n}\right\}$.

In this case use the definition of the limit.

Let $R < 0$. Set $N = 1 - R$. If $n > N$, then $\frac{1-n^2}{n} = \frac{1}{n} - n \leq 1 - n < 1 - N = 1 - (1 - R) = R$.

Therefore $\lim_{n \to \infty} \frac{1-n^2}{n} = -\infty$. (By definition.)

(c) $\left\{\frac{3n^3+1}{2n^3+n^2}\right\}$.

Dividing top and bottom by $n^3$ and using the algebra of limits we see that the above limit is $\lim_{n \to \infty} \frac{3+\frac{1}{n}}{2+\frac{1}{n}+\frac{1}{n^2}} = \frac{3+\lim_{n \to \infty} \frac{1}{n}}{\lim_{n \to \infty} \left(2+\frac{1}{n}+\frac{1}{n^2}\right)} = \frac{3}{2}$.

5. Verify the following limits:

(a) $\lim_{n \to \infty} a^{1/n} = 1$, for $a > 0$.

$$\lim_{n \to \infty} \log(a^{1/n}) = \lim_{n \to \infty} \frac{1}{n} \log(a) = 0$$ and so $\lim_{n \to \infty} a^{1/n} = \lim_{n \to \infty} e^{\log(a^{1/n})} = e^0 = 1$.

(b) $\lim_{n \to \infty} n^{1/n} = 1$.

$$\lim_{n \to \infty} \log(n^{1/n}) = \lim_{n \to \infty} \frac{1}{n} \log n = \lim_{x \to \infty} \frac{\log x}{x} = \lim_{x \to \infty} \frac{1}{x} = 0,$$

where we used l’Hopital’s rule in the last. Therefore $\lim_{n \to \infty} n^{1/n} = \lim_{n \to \infty} e^{\log(n^{1/n})} = e^0 = 1$.
(c) \( \lim_{n \to \infty} (a^n + b^n)^{1/n} = \max(a, b) \), where the RHS is the maximum of \( a \) and \( b \).

Assume wlog that \( a \geq b \). Then \( a \leq (a^n + b^n)^{1/n} \leq (2a^n)^{1/n} = 2^{1/n}a \). By (a) we have \( \lim_{n \to \infty} 2^{1/n}a = a \). Therefore by the Squeeze Theorem we get \( \lim_{n \to \infty} (a^n + b^n)^{1/n} = a = \max(a, b) \).

6. (a) Prove that if \( 0 < a < 2 \), then \( a < \sqrt{2}a < 2 \).

\[ a = \sqrt{a} \sqrt{a} < \sqrt{a \sqrt{2}} = \sqrt{2a} < \sqrt{2 \cdot 2} = 2. \]

(b) Prove that the sequence \( \sqrt{2}, \sqrt{2 \sqrt{2}}, \sqrt{2 \sqrt{2 \sqrt{2}}}, \ldots \) converges.

\[ a_1 = \sqrt{2} \text{ and } a_{n+1} = f(a_n), \text{ where } f(x) = \sqrt{2x}. \]

We will prove by induction: \( (P_n) \ a_n \leq a_{n+1} \text{ and } 0 < a_n < 2 \).

This is trivial if \( n = 1 : 0 < \sqrt{2} < 2 \) and \( \sqrt{2} < \sqrt{2 \cdot 2^1/4} \).

Assume \( (P_n) \). Then since \( f \) is increasing \( a_n \leq a_{n+1} \) implies that \( a_{n+1} = f(a_n) \leq f(a_{n+1}) = a_{n+2} \). By (a) \( 0 < a_n < 2 \) implies that \( a_{n+1} = f(a_n) < 2 \). We have proved \( (P_{n+1}) \) and so by Mathematical Induction \( (P_n) \) holds for all natural numbers \( n \). This means \( \{a_n\} \) is an increasing sequence which is bounded above and so must converge.

(c) Evaluate the limit in (b). Let \( L = \lim a_n \). Then \( L = \lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} f(a_n) = f(L) \). So solving \( L = \sqrt{2L} \) we get \( L^2 = 2L \iff L(2-L) = 0 \iff L = 0 \) or \( 2 \). Since \( L \)

is the least upper bound of the \( \{a_n\} \) we must have \( L \geq \sqrt{2} \) and so we must have \( L = 2 \).

7. Assume \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = L \) and \( a_n \leq c_n \leq b_n \) for \( n \geq n_0 \). Prove that \( \lim_{n \to \infty} c_n = L \). (That is, prove the Squeeze Theorem, which we have used many times.) Note here that proving that \( \{c_n\} \) is convergent is part of the question.

Let \( \varepsilon > 0 \). Choose \( N_1 \) so that \( n > N_1 \) implies \( |a_n - L| < \varepsilon \), and \( N_2 \) so that \( n > N_2 \) implies \( |b_n - L| < \varepsilon \). Let \( N = \max(N_1, N_2, n_0) \). If \( n > N \), then \( L - \varepsilon < a_n \leq c_n \leq b_n < L - \varepsilon \), and so \( |c_n - L| < \varepsilon \). This proves that \( \lim_{n \to \infty} c_n = L \).

8. Assume \( \lim_{x \to \infty} f(x) = L \) and \( \{a_n\} \) is a sequence in Domain(f), so that \( \lim_{n \to \infty} a_n = \infty \). Use the definitions of these limits to prove that \( \lim_{n \to \infty} f(a_n) = L \).

Let \( \varepsilon > 0 \). Choose \( R > 0 \) so that \( x > R \) implies \( |f(x) - L| < \varepsilon \). Now choose \( N \) so that \( n > N \) implies \( a_n > R \). This in turn implies (by the choice of \( R \)) that \( |f(a_n) - L| < \varepsilon \). Therefore \( \lim_{n \to \infty} f(a_n) = L \).