Math 121 Assignment 8 Due Wed. March 11 at start of class

You may use any results stated in class (except when asked to prove such a result as in Q2!).

1. Verify the following limits (a and b are positive):
   
   (a) \( \lim_{n \to \infty} a^{1/n} = 1. \)
   
   (b) \( \lim_{n \to \infty} (a^n + b^n)^{1/n} = \max(a, b), \) where the RHS is the maximum of \( a \) and \( b. \)

2. Assume that \( \ell_n \leq r_n \) ultimately and \( \ell_n \to \infty. \) Prove that \( r_n \to \infty. \)

3. (a) Prove that if \( 0 < a < 2, \) then \( a < \sqrt{2a} < 2. \)
   
   (b) Prove that the sequence \( \sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots \) converges.
   
   (c) Evaluate the limit in (b).

4. Find the sum of the given series or show that the series diverges:
   
   (a) \( \sum_{n=0}^{\infty} \frac{5}{10^{3n+1}}. \)
   
   (b) \( \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}. \)
   
   (c) \( \sum_{n=1}^{\infty} \frac{n}{n+2}. \)

5. A sequence \( \{x_n\} \) is Cauchy iff for any \( \varepsilon > 0 \) there is an \( N \) so that if \( m, n > N, \) then \( |x_n - x_m| < \varepsilon. \) Prove that a convergent sequence is Cauchy.

Remark. The converse is also true but you needn’t prove this harder result. It depends on the Completeness Axiom and is in fact equivalent to it. This characterization of convergence is useful since it does not depend on guessing a limit; it depends only on the given sequence.

6. (a) Prove that \( \frac{1}{n+1} \leq \log(n+1) - \log n \leq \frac{1}{n}. \)
   
   (b) If \( a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n, \) show that \( \{a_n\} \) is convergent.

   Hint. Show that \( \{a_n\} \) is decreasing.

7. Practice: Ch. 9. Sec. 9.1 #31, p. 567 #2