Math 121 Assignment 6 Due Wed. Feb. 27 [Note date] at start of class

You may use any results stated in class (except when asked to prove such a result!). This assignment has 2 pages. Questions 1-4 and the practice questions are recommended preparation for the midterm.

1. Decide if the following integrals converge or diverge to $\infty$. Justify your answers.
   
   (a) $\int_0^\infty e^{-x^4} \, dx$
   
   (b) $\int_0^\infty \frac{1-\cos x}{x^{5/2}} \, dx$. Hint: One approach is to use $1 - \cos x = \int_0^x \sin u \, du$ to estimate the integrand for $x \in [0, 1]$.

2. Evaluate (if possible): $\int_{-\pi/2}^{\pi/2} \log(\cos x) \tan x \, dx$.

3. Ex. 6.6 p. 378 #7
   
   Answer: $T_4 = 3000$ km$^2$, $T_8 = 3400$ km$^2$.

4. Assume $f : [a, b] \to \mathbb{R}$, $f''$ is continuous on $[a, b]$, and $f(a) = f(b) = 0$. Prove that
   
   $\int_a^b (b - x)(x - a)f''(x) \, dx = -2 \int_a^b f(x) \, dx$.
   
   Hint: One approach is to integrate by parts.

5. The gamma function $\Gamma(x)$ is defined by the improper integral
   
   $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t} \, dt$.
   
   (a) Show that the improper integral converges if $x > 0$.
   
   (b) Use integration parts to show that $\Gamma(x + 1) = x \Gamma(x)$ for all $x > 0$.
   
   (c) Show that $\Gamma(n + 1) = n!$ for all $n \in \mathbb{Z}_+ = \{0, 1, \ldots\}$.
   
   (d) Use the fact that $\int_{-\infty}^\infty \frac{e^{-x^2/2}}{\sqrt{2\pi}} \, dx = 1$ to show that $\Gamma(1/2) = \sqrt{\pi}$.

6. Assume $f : (a, b] \to \mathbb{R}$ is bounded and also is integrable on $[c, b]$ for every $c \in (a, b)$. Extend $f$ to $[a, b]$ by defining $f(a) = 42$. Prove that $f$ is integrable on $[a, b]$.
   
   Hint. One approach is to use the integrability test.
   
   This shows that for such functions there is no need to define the integral on $[a, b]$ as an improper integral.

7. Let $f : [a, \infty) \to \mathbb{R}$ be integrable on $[a, R]$ for all $R > a$. Let $f^+(x) = \max(f(x), 0)$ and $f^-(x) = \max(-f(x), 0)$, so that $f^\pm : [a, \infty) \to [0, \infty)$. 
(a) Show that $f = f^+ - f^-$ and $|f| = f^+ + f^-$. 

(b) Show that $f^+$ and $f^-$ are integrable on $[a,R]$ for all $R > a$. 

(c) Prove that

$$\int_a^\infty |f|\,dx \text{ is convergent } \iff \int_a^\infty f^+\,dx \text{ and } \int_a^\infty f^-\,dx \text{ are both convergent} \iff \int_a^\infty f\,dx \text{ is convergent.}$$

(d) Show the improper integral $\int_\pi^\infty \frac{\sin x}{x}\,dx$ is convergent. (One approach is to integrate by parts.) In this case one can show that $\int_\pi^\infty \frac{\sin x}{x}\,dx = \infty$ (see (e)), so one cannot directly use (c)–hence we see that the converse to (c) is in fact false.

(e) (Bonus question). Prove $\int_\pi^\infty \frac{\sin^2 x}{x}\,dx = \infty$. (Guess what method may help here.) Conclude that $\int_\pi^\infty \frac{\sin x}{x}\,dx = \infty$.

8. Practice questions (not to hand in). Section 6.5. #33, 37, 40