Solutions to Assignment 5

1. Evaluate (simplify your answers when possible):

(a) \[ \int \frac{x^2}{(1-x^2)^{5/2}} \, dx. \]

(b) \[ \int_3^4 \frac{dx}{x\sqrt{x^2-4}}. \]

(c) \[ \int \frac{dx}{3+2\sin x}. \]

(d) \[ \int \sqrt{x-x^2} \, dx. \]

(e) \[ \int_0^1 \frac{x^{1/2}}{1+x^{3/2}} \, dx. \]

Solutions

(a) Let \( x = 2 \sin \theta \) so that \( dx = 2 \cos \theta \, d\theta \) and

\[
\int \frac{x^2}{(4-x^2)^{5/2}} \, dx = \int \frac{4 \sin^2 \theta \cos \theta}{(4 \cos^2 \theta)^{5/2}} \, d\theta \\
= 8(4^{-5/2}) \int \sin^2 \theta \cos^{-4} \theta \, d\theta \\
= 4^{-1} \int \tan^2 \theta \sec^2 \theta \, d\theta \\
= 4^{-1} \int u^2 \, du \quad \text{where} \quad u = \tan \theta \\
= 4^{-1} \tan^3 \theta/3 = (12)^{-1} \tan^3(\arcsin(x/2)) = (12)^{-1} \left[ \frac{x}{\sqrt{4-x^2}} \right]^3 + C.
\]

The last inequality can be seen by drawing a right angle triangle with angle \( \theta \) satisfying \( \sin \theta = x/2 \) so that the sides can be \( \text{opp} = x, \ \text{hyp} = 2, \ \text{adj} = \sqrt{4-x^2}. \)

(b) \[ I = \int_3^4 \frac{dx}{x\sqrt{x^2-4}}. \] Let \( x = 2 \sec \theta \) so that \( dx = 2 \sec \theta \tan \theta \, d\theta \) and

\[
I = \int_{\sec^{-1}\frac{3}{2}}^{\sec^{-1}\frac{3}{2}} \frac{2 \sec \theta \tan \theta}{(2 \sec \theta)(2 \tan \theta)} \, d\theta = \frac{1}{2} (\sec^{-1}(2) - \sec^{-1}(3/2)) = \frac{\pi}{6} - \frac{1}{2} \sec^{-1}(3/2).
\]

(c) Let \( y = \tan(x/2) \) so that \( dx = \frac{2 \, dy}{1+y^2} \) and \( \sin x = \frac{2y}{1+y^2}. \) Therefore

\[
\int \frac{dx}{3+2\sin x} = \int \frac{1}{3 + \frac{4y}{1+y^2}} \, dy \\
= 2 \int \frac{1}{3 + 4y + 3y^2} \, dy \\
= 2 \left( \frac{3}{3} \int \frac{1}{(y+(2/3))^2 + (5/9)} \, dy \right) \\
= \frac{2}{3} \arctan \left( \frac{y+(2/3)}{\sqrt{5}/3} \right) + C \\
= \frac{2}{\sqrt{5}} \arctan \left( \frac{3y+2}{\sqrt{5}} \right) + C.
\]
Evaluate the following improper integrals:

(a) \( I = \int \frac{1}{\sqrt{1 - \frac{1}{4}x}} \, dx \) = \( \int \frac{1}{\sqrt{x - x^2}} \, dx \). Let \( x - \frac{1}{2} = \frac{1}{2} \sin \theta \), so that \( dx = \frac{1}{2} \cos \theta \, d\theta \) and
\[
I = \int \frac{1}{4} \cos^2 \theta \, d\theta = \frac{1}{8} \int 1 + \cos 2\theta \, d\theta = \frac{\theta}{8} \quad \text{and} \quad \theta = \frac{\theta}{16} \sin 2\theta / \sin \theta = \theta c / 8 + (\sin \theta \cos \theta) / 8
\]
\[
= \sin^{-1}(2x - 1)/8 + \frac{1}{2}(2x - 1)\sqrt{x - x^2} + C.
\]

(b) \( I = \int \frac{x^{1/2}}{1 + x^{3/2}} \, dx \). Let \( x = u^6 \) so that \( dx = 6u^5 \, du \) and (long division)
\[
I = \int \frac{6u^8}{1 + u^2} \, du = \int u^6 - u^4 + u^2 - 1 + \frac{1}{1 + u^2} \, du = 6\left(\frac{u^7}{7} - \frac{u^5}{5} + \frac{u^3}{3} - u + \arctan u\right)_{0}^{1}
\]
\[
= 6\left(\frac{1}{7} - \frac{1}{5} + \frac{1}{3} - 1 + \arctan 1\right) = \frac{3\pi}{2} - \frac{152}{35}.
\]

2. Evaluate the following improper integrals:

(a) \( \int_{-\infty}^{\infty} \frac{dx}{x^2 + 6x + 12} \).

(b) \( \int_{1/2}^{1} \frac{1}{\sqrt{x(1-x)}} \, dx \).

(c) \( \int_{0}^{\pi/2} \frac{\sec^2 x \, dx}{dx} \).

**Solutions**

(a) If \( w = x + 3 \), then \( \int (x^2 + 6x + 12)^{-1/2} \, dx = \int ((x+3)^2+3)^{-1/2} \, dx = \int (w^2 + 3) \, dw = 3^{-1/2} \arctan((x+3)/\sqrt{3}) + C \). Therefore
\[
\int_{-\infty}^{\infty} \frac{dx}{x^2 + 6x + 12} = \lim_{R \to \infty} 3^{-1/2} \arctan((x+3)/\sqrt{3})_{0}^{1} + \lim_{R \to \infty} 3^{-1/2} \arctan((x+3)/\sqrt{3})_{1}^{R}
\]
\[
= -(-3^{-1/2}\pi/2) + 3^{-1/2}\pi/2 = \pi/\sqrt{3}.
\]

Note the values of \( \arctan((x+3)/\sqrt{3}) \) at \( x = 0 \) cancel in the above.

(b) \( \int_{1/2}^{1} \frac{1}{\sqrt{x(1-x)}} \, dx = \int_{1/2}^{1} \frac{1}{\sqrt{4x} - 2x^2} \, dx = \arcsin(2x - 1) + C \). Therefore
\[
\int_{1/2}^{1} \frac{1}{\sqrt{x(1-x)}} \, dx = \lim_{c \to 1^-} \arcsin(2c - 1)_{1/2}^{c}
\]
\[
= \lim_{c \to 1^-} \arcsin(2c - 1) = \pi/2.
\]

(c) The limit argument in this question requires care. Take a close look at the argument.

Lemma: \( \lim_{c \to \pi/2^-} \log(\cos c) / \sec c = 0 \).

Pf. By L’Hospital the above limit is \( \lim_{c \to \pi/2^-} \frac{-\tan c}{\sec c \tan c} = \lim_{c \to \pi/2^-} \cos c = 0 \).

We showed in class that \( \int x \sec^2 x \, dx = x \tan x - \log(\sec x) + C \). Therefore
\[
\int_{0}^{\pi/2} x \sec^2 x \, dx = \lim_{c \to \pi/2^-} x \tan x - \log|\sec x|_{0}^{c}
\]
\[
= \lim_{c \to \pi/2^-} c \tan c - \log(\sec c)
\]
\[
= \lim_{c \to \pi/2^-} \sec c \left[ c \sin c + \log(\cos c) / \sec c \right]. \quad (1)
\]
By the Lemma the term in square brackets converges to \( \pi/2 > 1 \) as \( c \to \pi/2^- \). If \( M > 0 \) we may choose \( 0 < c_0 < \pi/2 \) so that if \( c \in (c_0, \pi/2) \), then \( \sec c > M \) and the term in square brackets in (1) is at least one. Therefore

\[
\sec c \left[ c \sin c + \log(\cos c)/\sec c \right] > M
\]

and it follows by definition that (1) is \( \infty \).

3. Find all \( p > 0 \) such that the improper integral

\[
I_p = \int_0^{e^{-2}} \frac{dx}{x|\log x|^p}
\]

converges. Justify your answer. When the integral converges, evaluate it.

**Solution.** If, for \( x \in (0, e^{-2}) \), \( u = |\log x| = \log(1/x) \), then \( du = -dx/x \) and we get (the improper integral is of course a limit which justifies the new limits on \( u \)),

\[
\int_0^{e^{-2}} \frac{dx}{x|\log x|^p} = \int_{2}^{\infty} u^{-p}du.
\]

We know the above \( p \)-integral converges iff \( p > 1 \) and hence so does the original improper integral. In this case we see the required integral is

\[
\left. \frac{u^{1-p}}{1-p} \right|_2^\infty = \frac{2^{1-p}}{p-1}.
\]

4. Assume \( f : [a, b] \to \mathbb{R} \), \( f'' \) is continuous on \([a, b]\), and \( f(a) = f(b) = 0 \). Prove that

\[
\int_a^b (b-x)(x-a)f''(x) \, dx = -2 \int_a^b f(x) \, dx.
\]

**Hint:** One approach is to integrate by parts.

**Solution** Let \( U = (b-x)(x-a) \) and \( dV = f''(x)dx \). Then \( dU = (a+b-2x)dx \) and \( V = f' \). So IBP shows the left-hand side equals

\[
(b-x)(x-a)f'(x)|_a^b - \int_a^b (a+b-2x)f'(x)dx = -\int_a^b (a+b-2x)f'(x)dx.
\]

IBP one more time with \( U = a+b-2x \) and \( dV = f'(x)dx \), so that \( dU = -2dx \) and \( V = f \). Therefore the above equals (there are 3 negative signs for the second term)

\[
-(a+b-2x)f(x)|_a^b - 2 \int_a^b f(x) \, dx = -2 \int_a^b f(x) \, dx,
\]

where in the last equality we use \( f(a) = f(b) = 0 \). This is the right-hand side of the desired equality, and so completes the argument.

5. Evaluate or prove the integral is not defined: \( \int_{-\pi/2}^{\pi/2} \log(\cos x) \tan x \, dx \).

**Solution.** The integrand is a continuous odd function on \((-\pi/2, \pi/2)\) but converges to \( \infty \) as \( x \to (-\pi/2)^+ \) and to \( -\infty \) as \( x \to \pi/2^- \). So it is
an improper integral. Recall from HW4 Q1(b) that \( \int \log(\cos x) \tan x \, dx = -\frac{1}{2} [\log(\cos x)]^2 \). Therefore

\[
\int_{-\pi/2}^{0} \log(\cos x) \tan x \, dx = \lim_{c \to (-\pi/2)^+} \int_{c}^{0} \log(\cos x) \tan x \, dx
\]

\[
= \lim_{c \to (-\pi/2)^+} 0 + \log(c)^2/2 = \infty.
\]

Similarly we get

\[
\int_{0}^{\pi/2} \log(\cos x) \tan x \, dx = \lim_{c \to \pi/2^-} \int_{0}^{c} \log(\cos x) \tan x \, dx
\]

\[
= \lim_{c \to \pi/2^-} -[\log(\cos c)]^2/2 + 0
\]

\[
= -\infty.
\]

Therefore \( \int_{-\pi/2}^{\pi/2} \log(\cos x) \tan x \, dx \) leads to the indeterminate form \( \infty - \infty \) and so is not defined.

6. Assume \( f : (a, b] \to \mathbb{R} \) is bounded and also is integrable on \( [c, b] \) for every \( c \in (a, b) \). Extend \( f \) to \([a, b] \) by defining \( f(a) = y_0 \) for some constant \( y_0 \).

(a) Prove that \( f \) is integrable on \([a, b] \).

(b) Prove that \( \int_{a}^{b} f \, dx \) does not depend on the choice of \( y_0 \).

**Hint.** One approach to (a) is to use the integrability test.

This shows that for functions such as \( f \) above, there is no need to define the integral on \([a, b] \) as an improper integral!

**Solution.** (a) Assume \( M > 1 \) satisfies \( |f(x)| \leq M \) for all \( x \in [a, b] \). Let \( \varepsilon > 0 \). Choose \( 0 < \delta < \min\left(\frac{\varepsilon}{4M}, b - a \right) \). By integrability of \( f \) on \([a + \delta, b] \) (an interval of positive length by the choice of \( \delta \)), there is a partition, \( Q \), of \([a + \delta, b] \) so that \( U(f, Q) - L(f, Q) < \frac{\varepsilon}{2} \). Let \( P = Q \cup \{a\} \), a partition of \([a, b] \).

Let \( M_i, m_i \) be the usual sup and inf of \( f \) over the \( i \)th interval in \( P \). Then as \(-M\) is a lower bound of \( f \) and \( M \) is an upper bound of \( f \) over \([a, b] \), it follows that

\[
m_i, M_i \in [-M, M] \quad \text{for all} \ i.
\]

Therefore we have

\[
U(f, P) - L(f, P) = (M_1 - m_1)\delta + (U(f, Q) - L(f, Q)) < 2M\delta + \frac{\varepsilon}{2} \quad \text{(by (2) and our choice of} Q)\]

\[
< 2M \frac{\varepsilon}{4M} + \frac{\varepsilon}{2} = \varepsilon.
\]

The integrability test now shows that \( f \) is integrable over \([a, b] \).

(b) Define \( g(x) = f(x) \) for \( x \in (a, b] \) and \( g(a) = 0 \). So \( g \) is integrable on \([a, b] \)
by taking \( y_0 = 0 \) in (a). It clearly suffices to prove that

\[
\int_a^b f dx = \int_a^b g dx,
\]

because the RHS does not depend on the choice of \( y_0 \). Recall \( I_a(x) \) equals 1 if \( x = a \) and is zero elsewhere and that (e.g. Q4 on HW 3) \( \int_a^b I_a dx = 0 \). We have \( f = g + y_0 I_a \) and so by linearity of the integral,

\[
\int_a^b f dx = \int_a^b g + y_0 I_a dx = \int_a^b g dx + y_0 \int_a^b I_a dx = \int_a^b g dx.
\]

This proves (3) and the proof is complete.

7. Practice questions (not to hand in). Sec. 6.5 #6, 11, 18, 22