Math 121 Solutions to Assignment 4

1. Evaluate (substitutions):

(a) \( \int \sec^3 x \tan^3 x \, dx \).

(b) \( \int \log(\cos x) \tan x \, dx \).

**Solutions.**

(a) Let \( u = \sec x \) so that \( du = \sec x \tan x \, dx \) and therefore

\[
\int \sec^3 x \tan^3 x \, dx = \int \sec^2 x (1 - \sec^2 x)(\sec x \tan x) \, dx = \int u^2 (u^2 - 1) \, du = \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C.
\]

(b) Let \( u = \log(\cos x) \), so that \( du = -(\sin x / \cos x) \, dx \). Therefore

\[
\int \log(\cos x) \tan x \, dx = \int \log(\cos x) \frac{\sin x}{\cos x} \, dx = -\int u \, du = -\frac{1}{2} u^2 + C = -\frac{1}{2} [\log(\cos x)]^2 + C.
\]

2. Evaluate (integration by parts):

(a) \( \int (\log x)^2 \, dx \).

(b) \( \int x \sec x \tan x \, dx \).

**Solutions.**

(a) Let \( U = (\log x)^2 \) and \( dV = dx \) so that \( dU = \frac{2 \log x \, dx}{x} \) and \( V = x \). IBP gives

\[
\int (\log x)^2 \, dx = x(\log x)^2 - 2 \int x \frac{\log x \, dx}{x} = x(\log x)^2 - 2 \int \log x \, dx = x(\log x)^2 - 2(\log x - x) + C = x(\log x)^2 - 2x \log x + 2x + C.
\]

(b) Let \( U = x \) and \( dV = \sec x \tan x \, dx \) so that \( dU = dx \) and \( V = \int \sec x \tan x \, dx = \sec x \). So IBP gives

\[
\int x \sec x \tan x \, dx = x \sec x - \int \sec x \, dx = x \sec x - \log |\sec x + \tan x| + C.
\]

3. Evaluate:

(a) \( \int_0^1 x^5 e^{x^2} \, dx \).

(b) \( \int \log(\sqrt{1 + x^2}) \, dx \).

(c) \( \int_0^1 \sqrt{x} \sin(\pi \sqrt{x}) \, dx \).

(d) \( \int_0^4 \sqrt{x} e^{\sqrt{x}} \, dx \).

**Solution.**

(a). The substitution \( u = x^2 \), so that \( du/2 = x \, dx \), leads to

\[
I := \int_0^1 x^5 e^{x^2} \, dx = \frac{1}{2} \int_0^1 u^2 e^u \, du.
\]

Now integrate by parts twice to conclude

\[
I = \frac{1}{2} [u^2 e^u]_0^1 - 2 \int_0^1 u e^u \, du = \frac{e}{2} - \int_0^1 ue^u \, du
\]

\[
= \frac{e}{2} - [ue^u]_0^1 - \int_0^1 e^u \, du = \frac{e}{2} - e + e - 1 = \frac{e}{2} - 1.
\]

(b) \( I = \int \log(\sqrt{1 + x^2}) \, dx = \frac{1}{2} \int \log(1 + x^2) \, dx \). Now set \( U = \log(1 + x^2) \) and \( dV = dx \), so that \( dU = \frac{2x}{1+x^2} \, dx \) and \( V = x \). Integrating by parts gives
6. Evaluate:

\[ I = \frac{1}{2} \left[ x \log(1 + x^2) - 2 \int \frac{x^2}{1 + x^2} dx \right] = \frac{x \log(1 + x^2)}{2} - \int 1 - \frac{1}{1 + x^2} dx = \frac{x \log(1 + x^2)}{2} - x + \arctan x + C. \]

(c) \[ I = \int_0^1 \sqrt{x} \sin(\pi \sqrt{x}) dx. \] Let \( u = \sqrt{x} \) so \( x = u^2 \) and \( dx = 2udu \). Then \[ I = \int_0^1 u \sin(\pi u)2udu = \frac{2}{\pi} \int_0^\pi w^2 \sin(w) dw. \] Let \( U = w^2 \) and \( dV = \sin w \), so that \( dU = 2wdw \) and \( V = -\cos w \). Therefore doing this and then integrating by parts once more leads to \[
I = \frac{2}{\pi} \left[ -w^2 \cos w \right]_0^\pi + 2 \int_0^\pi w \cos wdw = \frac{2}{\pi^3} \left[ \pi^2 + 2w \sin w \right]_0^\pi - 2 \int_0^\pi \sin wdw = \frac{2}{\pi^3} [\pi^2 - 4] = 2/\pi - 8/\pi^3.
\]

(d) \[ I = \int_0^4 4x e^{\sqrt{x}} dx. \] Set \( x = w^2 \) (note \( x \geq 0 \)) so that \( dx = 2wdw \) and then integrating by parts below we have
\[
I = \int_0^2 w e^{w}2wdw = 2 \int_0^2 w^2 e^w dw = 2[w^2 e^w]_0^2 - 2 \int_0^2 w e^w dw = 2(4e^2 - 4e^2 + 2(e^2 - 1)) = 4(e^2 - 1).
\]

4. Obtain a reduction formula for \( I_n = \int \tan^n x dx \) and use it to evaluate \( \int \tan^7 x dx \).

**Solution.** \( I_0 = x \) and \( I_1 = \log(|\sec x|) + C \) as noted in class. For \( n \geq 2 \) use the substitution \( u = \tan x \) to see
\[
I_n = \int \tan^{n-2} x (\sec^2 x - 1) dx = \int u^{n-2} du - I_{n-2} = \frac{\tan^{n-1} x}{n-1} - I_{n-2}.
\]
So \( \int \tan^7 x dx = \int \tan x 6 - I_5 = \int \tan x 6 - \tan^4 x 4 + I_3 = \int \tan x 6 - \tan^4 x 4 + \tan^2 x 2 - \log(|\sec x|) + C. \)

5. Find the areas of the following regions:

(a) The bounded region enclosed by the curve \( y^2 = x^6(1 - x^4) \).

(b) The bounded region bounded by \( y = x^4 \) and \( y = 6 - x^2 \).

**Solution.** (a) As \( x^6(1 - x^4) \) must be positive we see that \( x \) is confined to \([-1, 1]\). It is the region bounded between the graphs \( y = \pm |x|^3 \sqrt{1 - x^4} \) for \( x \in [-1, 1] \). By symmetry this area is four times the area of the region in the first quadrant and so equals \( I = 4 \int_0^1 x^3 \sqrt{1 - x^4} dx \). Letting \( u = 1 - x^4 \) so that \( du = -4x^3dx \) we arrive at
\[
I = \int_1^0 -\sqrt{u} du = \int_0^1 \sqrt{u} du = \frac{2}{3} u^{3/2} \bigg|_0^1 = 2/3.
\]

(b) We first find the intersection points of the graphs \( y = x^4 \) and \( y = 6 - x^2 \) by setting \( x^4 = 6 - x^2 \). This holds iff \( x^4 + x^2 - 6 = 0 \) which holds iff \((x^2 + 3)(x^2 - 2) = 0 \) and so the roots are \( x = \pm \sqrt{2} \). Drawing a picture (you should!) one can see that \( 6 - x^2 \geq x^4 \) on \([-\sqrt{2}, \sqrt{2}] \) and so the area of the region equals
\[
\int_{-\sqrt{2}}^{\sqrt{2}} 6 - x^2 - x^4 dx = 2 \int_{\sqrt{2}}^{\sqrt{2}} 6 - x^2 - x^4 dx = 2 \left[ 6\sqrt{2} - \left( \frac{\sqrt{2}}{3} \right)^3 - \left( \frac{\sqrt{2}}{5} \right)^5 \right] = \frac{136}{15} \sqrt{2}.
\]

6. Evaluate:

(a) \( \int \frac{1}{x^{1+1}} dx \).

(b) \( \int \frac{2x}{(x^2 + x + 1)^2} dx \).
(c) \( \int \frac{e^{x+1}}{e^{x+1}} \, dx \).

**Solutions.** (a) \( I = \int \frac{1}{x^2-1} \, dx \). Seek \( A, B, C, D \) s.t. \( \frac{1}{(x-1)(x+1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x^2+1} + \frac{Cx+D}{x^2+1} \). This easily is solved by \( A = 1/4, B = -1/4, C = 0 \) and \( D = -1/2 \). Therefore

\[
I = \int \left( \frac{1}{4x-4} - \frac{1}{4x+4} - \frac{1}{2} - \frac{1}{2x^2+1} \right) \, dx = \frac{1}{4} \log |x-1| - \frac{1}{4} \log |x+1| - \frac{1}{2} \arctan x + C.
\]

(b) \( I := \int \frac{2x}{(x^2+1)^2} \, dx = \int \frac{2x+1}{(x^2+1)^2} \, dx - \int \frac{1}{(x^2+1)^2} = J_1 - J_2. \)

For \( J_1 \) let \( u = x^2 + x + 1 \) so that \( du = (2x+1) \, dx \) and we have

\[
J_1 = \int u^{-2} \, du = -u^{-1} + C = -(x^2 + x + 1)^{-1} + C.
\]

For \( J_2 \) we complete the square, let \( w = x + (1/2) \), and use the formula for \( \int (w^2 + a^2)^{-2} \, dw \) derived in class from the quadratic reduction formula \((QR_n)\) with \( n = 2 \):

\[
J_2 = \int \frac{1}{((x + (1/2))^2 + (3/4))^2} \, dx
= \int (w^2 + (3/4)^2)^{-2} \, dw
= \frac{1}{2(3/4)} \left[ \frac{2}{\sqrt{3}/2} \arctan(w/(\sqrt{3}/2)) + \frac{w}{w^2 + (3/4)} \right] + C
= \frac{2}{3} \arctan((2x + 1)/\sqrt{3}) + \frac{x + (1/2)}{(x + (1/2)^2 + (3/4)} + C
= \frac{4}{3\sqrt{3}} \arctan((2x + 1)/\sqrt{3}) + \frac{2x + 1}{3(x^2 + x + 1)} + C.
\]

So combining the above, we get

\[
I = -(x^2 + x + 1)^{-1} - \frac{4}{3\sqrt{3}} \arctan((2x + 1)/\sqrt{3}) - \frac{2x + 1}{3(x^2 + x + 1)} + C.
\]

(c) \( I = \int \frac{e^{x+1}}{e^{x+1}} \, dx \). Let \( u = e^x > 0 \) so that \( x = \log u \) and \( dx = du/u \). Then

\[
I = \int \frac{u+1}{u(u-1)} \, du.
\]

Set \( \frac{u+1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1} \). This easily gives \( A = -1 \) and \( B = 2 \) and so (recall \( u > 0 \))

\[
I = \int \frac{1}{u} \, du + 2 \int \frac{du}{u-1} = -\log |u| + 2 \log |u-1| + C = -x + 2 \log (e^x - 1) + C.
\]

7. Let \( f \) be uniformly continuous on \((0, 1)\). Prove \( f \) is bounded on \((0, 1)\).

**Solution.** By uniform continuity of \( f \) with \( \varepsilon = 1 \), there is a \( \delta > 0 \) s.t. \( \forall x, x' \in (0, 1) \) if \( |x - x'| < \delta \), then \( |f(x) - f(x')| < 1 \). Let \( P = \{x_0, x_1, \ldots, x_N\} \) be a partition of \([0, 1]\) such that \( \|P\| < \delta \). (E.g. take \( N \) s. t. \( 1/N < \delta \) and set \( x_i = i/N \).) Let \( M = \max \{|f(x_i)| : i = 1, \ldots, N - 1\} \in \mathbb{R}_+ \). We claim that

\[
\forall x \in (0, 1) \quad |f(x)| \leq M + 1. \tag{1}
\]

The result would then clearly follow. To prove (1), let \( x \in (0, 1) \). We may choose \( i \in \{1, \ldots, N - 1\} \) s. t. \( |x - x_i| < \delta \). (For this, note that either \( x \in (x_{i-1}, x_i) \) for some \( i \in \{1, \ldots, N - 1\} \) or \( x \in [x_{N-1}, x_N) \)). The above \( \delta \) bound implies that \( |f(x_i) - f(x)| < 1 \), and therefore by the triangle inequality,

\[
|f(x)| \leq |f(x_i)| + |f(x_i) - f(x)| < M + 1. \] This proves (1). \( \square \)
8. Practice questions (not to hand in).
   Sec. 5.7 #15,30; Sec. 6.1 p.339-340. #10, 20, 31, 36