Math 121 Solutions to Assignment 2 Due Wed. Jan. 16 at start of class

You may use any theorems stated in class.

1. Express the following limits as definite integrals (justify your answers).

(a) \( \lim_{n \to \infty} \sum_{i=1}^{n} \frac{n}{n^2 + i^2} \)

(b) \( \lim_{n \to \infty} \sum_{i=1}^{n} i^{1/3} n^{-4/3} \).

**Solution.** (a) Dividing top and bottom by \( n^2 \) we see the limit is

\[
\lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{1 + (i/n)^2} = \int_{0}^{1} \frac{1}{1 + x^2} \, dx.
\]

Here we have used the continuity of \( f(x) = (1 + x^2)^{-1} \) to apply Theorem 5.13 from class which says the integral is the limit of a sequence of Riemann sums for partitions whose norms approach zero.

(b) \( \lim_{n \to \infty} \sum_{i=1}^{n} i^{1/3} n^{-4/3} = \lim_{n \to \infty} \sum_{i=1}^{n} \left( \frac{i}{n} \right)^{1/3} \frac{1}{n} = \int_{0}^{1} x^{1/3} \, dx. \)

Again we are using the continuity of \( f(x) = x^{1/3} \) as in (a).

2. Assume \( a < c < b \). If \( f \) is integrable on \([a, c]\) and on \([c, b]\), prove that \( f \) is integrable on \([a, b]\). (Note: To be precise one should say the restrictions of \( f \) to \([a, c]\) and \([c, b]\) are integrable on these intervals.)

**Solution.** Let \( \varepsilon > 0 \). Choose partitions \( P_1 \) of \([a, c]\) and \( P_2 \) of \([c, b]\) so that \( (U - L)(f, P_1) < \varepsilon/2 \) and \( (U - L)(f, P_2) < \varepsilon/2 \). (They exist by the Integrability Test.) Then \( P = P_1 \cup P_2 \) is a partition of \([a, b]\) and, as noted in class,

\[
(U - L)(f, P) = (U - L)(f, P_1) + (U - L)(f, P_2) < 2\varepsilon/2 = \varepsilon.
\]

By the Integrability Test, \( f \) is integrable on \([a, b]\).

3. A function \( f \) is increasing if \( x \leq x' \) implies \( f(x) \leq f(x') \), and is decreasing if \( x \leq x' \) implies \( f(x) \geq f(x') \). We say \( f \) is monotone if it is either increasing or decreasing. Let \( f : [a, b] \to \mathbb{R} \) be a monotone function.

(a) Prove that for any partition \( P = \{x_0, x_1, \ldots, x_n\} \) of \([a, b]\),

\[
U(f, P) - L(f, P) \leq |f(b) - f(a)||\|P\|. \]

(Recall that \( \|P\| = \max_{i=1}^{n} \Delta x_i \).

(b) Prove that \( f \) is integrable on \([a, b]\).
(c) Use (a) and (b) to show that for any partition $P$ of $[a, b]$,

$$\left| \int_a^b f(x) \, dx - L(f, P) \right| \leq |f(b) - f(a)||P|$$

and

$$\left| \int_a^b f(x) \, dx - U(f, P) \right| \leq |f(b) - f(a)||P|$$

Solution. (a) Let’s assume $f$ is increasing and let $P = \{x_0, \ldots, x_n\}$. Since $f$ is increasing we have $M_i = f(x_i)$ and $m_i = f(x_{i-1})$. Therefore

$$U(f, P) - L(f, P) = \sum_{i=1}^n (f(x_i) - f(x_{i-1})) \Delta x_i$$

$$\leq \max\{\Delta x_i : i = 1, \ldots, n\} \sum_{i=1}^n f(x_i) - f(x_{i-1}) = ||P|| (f(b) - f(a)).$$

In the last line we used the fact that we have a telescoping sum. The proof for $f$ decreasing is similar.

(b) We may assume $|f(b) - f(a)| > 0$, since otherwise $f$ is constant. Let $\epsilon > 0$. Choose $P$ so that $||P|| < \epsilon / |f(b) - f(a)|$. Then by (a) we have

$$U(f, P) - L(f, P) \leq ||P|| |f(b) - f(a)| < \epsilon.$$ 

By the Integrability Test, $f$ is integrable on $[a, b]$.

(c) By the definition of the Riemann integral and (b) we have,

$$0 \leq \int_a^b f(x) \, dx - L(f, P) \leq U(f, P) - L(f, P) \leq |f(b) - f(a)||P|.$$ 

This gives the first inequality and the second one is proved in the same way.

4. Prove that for any $a < b$,

(a) $(a, b) \cap \mathbb{Q}$ is non-empty.

**Hint.** Choose a natural number $n$ so that $1/n < b - a$. If $[x]$ denotes the unique integer $m$ so that $x \in [m-1, m)$ consider $r = [na]/n$.

(b) $(a, b) \cap \mathbb{Q}^c$ is non-empty.

**Hint.** One approach is to first show that $\sqrt{2}r$ is irrational for every rational $r$. (You may assume $\sqrt{2}$ is irrational.)

Solution. (a) Clearly $r = [na]/n$ is rational. Note that if $n$ is chosen as in the Hint,

$$a = \frac{na}{n} < \frac{[na]}{n} \leq \frac{na+1}{n} = a + \frac{1}{n} < a + (b-a) = b,$$

so that $r \in (a, b)$.

(b) Let $r \neq 0$ be rational. If $\sqrt{2}r = s$ is rational, then $\sqrt{2} = sr^{-1}$ is rational, a contradiction. Therefore $A = \{\sqrt{2}r : r \in \mathbb{Q}, r \neq 0\} \subseteq \mathbb{Q}^c$.

Now by (a) $(a/\sqrt{2}, b/\sqrt{2})$ contains a rational number $r$. If $r = 0$, then this interval will contain $1/n$ for large enough $n$ so that $1/n < b/\sqrt{2}$ and so we may assume $r \neq 0$. Therefore $a < \sqrt{2}r < b$ and so $(a, b)$ contains $\sqrt{2}r$ which is irrational by the above.
5. Let $f$ be a bounded function on $[a, b]$ and $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$. Define $M_i$ and $m_i$ as usual and let $M_i'$ and $m_i'$ have the usual meanings but for $|f|$.

(a) Prove that $M_i' - m_i' \leq M_i - m_i$.

**Hint:** One approach is to let $\varepsilon > 0$ and choose $y_i \in [x_{i-1}, x_i]$ such that $|f(y_i)| \geq M_i' - \varepsilon$. (Why is this possible?) You may also use the trivial fact that for any reals $A, B$, we have by the triangle inequality, $|A| - |B| \leq |A - B|$.

(b) Prove that if $f$ is integrable on $[a, b]$, then so is $|f|$.

**Hint:** Even if you don’t do (a), if you assume (a) this is fairly easy.

(c) Prove that for any real numbers $A, B$, $\max(A, B) = \frac{A + B + |A - B|}{2}$. Here $\max(A, B)$ is the maximum of $A$ and $B$, and, yes, this is very easy.

(d) Prove that if $f$ and $g$ are integrable on $[a, b]$, then so is $h(x) = \max(f(x), g(x))$.

**Solution.** (a) Let $\varepsilon > 0$. By HW1 Q4 we may choose $y_i \in [x_{i-1}, x_i]$ s.t. $|f(y_i)| \geq M_i' - \varepsilon$. Similar reasoning allows us to choose $z_i \in [x_{i-1}, x_i]$ s.t. $|f(z_i)| \leq m_i' + \varepsilon$. Assume wolog that $f(y_i) \geq f(z_i)$ (the argument is the same if the opposite inequality holds). Then

$$M_i' - m_i' \leq |f(y_i)| - |f(z_i)| + 2\varepsilon$$

$$\leq |f(y_i) - f(z_i)| + 2\varepsilon$$

$$= f(y_i) - f(z_i) + 2\varepsilon$$

$$\leq M_i - m_i + 2\varepsilon.$$

In the second inequality we have used the triangle inequality as noted in the question. As the above inequality holds for any $\varepsilon > 0$, the desired inequality follows.

(b) Let $\varepsilon > 0$ and use the Integrability Test (and the integrability of $f$) to find a partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$ s.t. $U(f, P) - L(f, P) < \varepsilon$. Now use (a) to conclude that

$$U(|f|, P) - L(|f|, P) = \sum_{i=1}^{n} (M_i' - m_i') \Delta x_i$$

$$\leq \sum_{i=1}^{n} (M_i - m_i) \Delta x_i = U(f, P) - L(f, P) < \varepsilon.$$

Another application of the Integrability Test now shows that $|f|$ is integrable on $[a, b]$.

(c) Assume wolog that $A \geq B$. Then $\frac{A + B + |A - B|}{2} = (A + B + A - B)/2 = A = \max(A, B)$.

(d) By (d), $(*)h(x) = (f(x) + g(x))/2 + |(f(x) - g(x))/2|$. Recall from the Linearity of the Integral that $(f + g)/2$ is integrable and
$(f - g)/2$ is integrable. By the last and (b), $|(f - g)/2|$ is also integrable. So (* ) shows $h$ is the sum of two integrable functions and hence is integrable by the Arithmetic of Integrals.

6. Practice Questions (Do not hand in): Sec. 5.4 Exercises (p. 312-313) #7, #9, #20, #38.