Math 121 Assignment 2 Due Wed. Jan. 22 at start of class

You may use results stated in class whenever reasonable (E.g., it is clearly not in question 5, which is a result stated in class!). This assignment has 2 pages.

1. Express the following limits as definite integrals (justify your answers). Note there may be more than one correct answer but you need only find one.

(a) \[ \lim_{n \to \infty} \sum_{j=1}^{n} \frac{j}{n^2} \sqrt{1 + \frac{j^2}{n^2}}. \]

(b) \[ \lim_{n \to \infty} \sum_{i=n}^{2n} (2^i)^{1/n} n^{-1}. \]

(c) \[ \lim_{n \to \infty} \sum_{i=1}^{n} (2i)^{2/3} n^{-5/3}. \]

2. Assume \( f : [a, b] \to \mathbb{R} \) is a monotone function and \( \{P_n\} \) is a sequence of partitions of \([a, b]\) such that \( \|P_n\| \to 0 \). Prove that \( \int_a^b f \, dx = \lim_{n \to \infty} L(f, P_n) = \lim_{n \to \infty} U(f, P_n) \). Hint: Recall Assignment 1.

Just as in our proof for continuous functions, it follows from the above that for monotone \( f \), \( \int_a^b f \, dx = \lim_{n \to \infty} R(f, P_n, c^n) \), where \( c^n \) is any choice vector for \( P_n \) (you need not repeat this argument though).

3. Assume \( a < c < b \). If \( f \) is integrable on \([a, b]\), prove that \( f \) is integrable on \([a, c]\) and on \([c, b]\). (Note: To be precise one should say the restrictions of \( f \) to \([a, c]\) and \([c, b]\) are integrable on these intervals.) In class we proved the converse to this statement.

4. Some elementary texts define the integral by \( I_0^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f\left(\frac{ib}{n}\right) \frac{b}{n} \).

We call this the HS integral of \( f \) on \([0, b]\). As we noted in class and in Q 2 above, for \( f \) continuous or monotone this agrees with the definition of the Riemann integral. But in general it gives some bad answers as we now show.

Let \( f : [0, 2] \to \mathbb{R} \) be 1 on the rationals and 0 on the irrationals. Evaluate \( I_0^1 f(x) \, dx \) and \( I_0^{\sqrt{2}} f(x) \, dx \) and hence show they exist. Conclude that \( I_0^1 f(x) \, dx > I_0^{\sqrt{2}} f(x) \, dx \), an absurd result for a non-negative function. [You may assume \( \sqrt{2} \) is irrational.]

5. Let \( f \) be integrable on \([a, b]\) and \( C \) be a real number. Prove that \( Cf \) is integrable on \([a, b]\) and \( \int_a^b Cf \, dx = C \int_a^b f \, dx \). Hint: Consider \( C = 0, C > 0 \) and \( C < 0 \) separately.

Note that the proof of the linearity of the integral online uses this result, so please do not just quote that result! (Note also we only proved this in class for \( f \) continuous.)
6. Evaluate (your answer may in terms of integrals):

(a) \( \frac{d}{dx} \left[ \int_{1}^{x} t^2 e^{-t^2} \, dt \right] \).

(b) \( \frac{d}{dx} \left[ \int_{1}^{x} x^2 e^{-t^2} \, dt \right] \).

7. (PRACTICE: NOT TO HAND IN) Let \( f \) be a bounded function on \([a, b]\) and \( P = \{x_0, \ldots, x_n\} \) be a partition of \([a, b]\). Define \( M_i \) and \( m_i \) as usual and let \( M'_i \) and \( m'_i \) have the usual meanings but for \( |f| \).

(a) Prove that \( M'_i - m'_i \leq M_i - m_i \).

**Hint:** One approach is to let \( \varepsilon > 0 \) and choose \( y_i \in [x_{i-1}, x_i] \) such that \( |f(y_i)| \geq M'_i - \varepsilon \). (Why is this possible?) You may also use the trivial fact that for any reals \( A, B \), we have by the triangle inequality, \( |A| - |B| \leq |A - B| \).

(b) Prove that if \( f \) is integrable on \([a, b]\), then so is \( |f| \).

**Hint:** Even if you don’t do (a), if you assume (a) this is fairly easy.

(c) Prove that for any real numbers \( A, B \), \( \max(A, B) = \frac{A+B+|A-B|}{2} \). Here \( \max(A, B) \) is the maximum of \( A \) and \( B \), and, yes, this is very easy.

(d) Prove that if \( f \) and \( g \) are integrable on \([a, b]\), then so is \( h(x) = \max(f(x), g(x)) \).

8. Practice Questions (Do not hand in): Sec. 5.4 Exercises (p. 312-313) #7, #9, #20, #38.