Math 121 Solutions to Assignment 1

1. Evaluate (show work as always): \( \sum_{i=2}^{100} \log \left( 1 + \frac{2}{i} \right) \).

Solution:
\[
\sum_{i=2}^{100} \log \left( 1 + \frac{2}{i} \right) = \sum_{i=2}^{100} \log(1 + 2) - \log i = \sum_{j=4}^{102} \log(j) - \sum_{i=2}^{100} \log i = \log(102) + \log(101) - \log(3) - \log(2) = \log(1717).
\]

2. For each of the following sets find (i) the sets \( \mathcal{L} \) and \( \mathcal{U} \) of lower bounds and upper bounds, respectively, (ii) the given set’s least upper bound and greatest lower bound, if they exist. You need not provide justifications.

(a) \( A = \left\{ \frac{3}{n^2} : n \in \mathbb{N} \right\} \)

(b) \( B = \{ x \in \mathbb{R} : x \text{ rational}, 0 < x^2 \leq 2 \} \)

(c) \( C = \{ x : x^2 - 3x + 2 < 0 \} \)

Solution: (a) \( A = \left\{ \frac{3}{n^2} : n \in \mathbb{N} \right\}, \mathcal{L} = (-\infty, 0], \mathcal{U} = [3, \infty) \). Therefore \( \sup(A) = \min \mathcal{U} = 3 \) and \( \inf(A) = \max \mathcal{L} = 0 \).

(b) \( B = \{ x \in \mathbb{R} : x \text{ rational}, 0 < x^2 \leq 2 \}, \mathcal{L} = (-\infty, -\sqrt{2}], \mathcal{U} = [\sqrt{2}, \infty) \) (a proof would use Question 5 below). Therefore \( \sup(A) = \min \mathcal{U} = \sqrt{2} \) and \( \inf(A) = \max \mathcal{L} = -\sqrt{2} \).

(c) \( C = \{ x : (x - 1)(x - 2) < 0 \} = (1, 2) \). So \( \mathcal{L} = (-\infty, 1], \mathcal{U} = [2, \infty), \sup(C) = 2 \) and \( \inf(C) = 1 \).

3. Let \( A \) be a non-empty subset of real numbers which is bounded below and \(-A = \{ -a : a \in A \} \).

(a) Prove that \( \sup(-A) \) exists.

(b) Prove that \( \inf A = -\sup(-A) \).

Solution: (a) If \( K \) is a lower bound for \( A \), then \(-K \) is an upper bound for \(-A \), and so \(-A \) is bounded above. Clearly it is also non-empty since \( A \) is. Therefore by the Completeness Axiom for the real numbers, \( \sup(-A) \) exists.

(b) If \( a \in A \), then \(-a \leq \sup(-A) \) and therefore \( a \geq -\sup(-A) \). We have shown \(-\sup(-A) \) is an lower bound for \( A \).

Let \( \ell \) be any lower bound for \( A \). If \( a \in A \) we have \( a \geq \ell \) and so \(-a \leq -\ell \). Therefore \(-\ell \) is an upper bound for \(-A \) and so \( \sup(-A) \leq -\ell \). The latter implies \( -\sup(-A) \geq \ell \) and so \( -\sup(-A) \) is the greatest lower bound of \( A \), as required.
4. Let \( A \) be a non-empty set bounded below and let \( \ell = \inf A \). Prove that for every \( \epsilon > 0 \), the set \([\ell, \ell + \epsilon) \cap A\) is non-empty. (Hint: this is an easy consequence of the definition.)

Of course a similar argument shows that if \( A \) is a non-empty set bounded above, and \( u = \sup A \), then for every \( \epsilon > 0 \), the set \((u - \epsilon, u) \cap A\) is non-empty (you need not show this). These are useful results to have at your fingertips.

**Solution:** Since \( \ell \) is the greatest lower bound of \( A \), \( \ell + \epsilon \) cannot be a lower bound for \( A \). So there must be an \( a \in A \) satisfying \( a < \ell + \epsilon \). As \( \ell \) is a lower bound for \( A \) we must also have \( a \geq \ell \). Therefore \( a \in [\ell, \ell + \epsilon) \) and so \([\ell, \ell + \epsilon) \cap A\) is non-empty.

5. Prove that for any \( a < b \),

(a) \( (a, b) \cap \mathbb{Q} \) is non-empty.

**Hint:** Choose a natural number \( n \) so that \( 1/n < b - a \). If \( \lfloor x \rfloor \) denotes the unique integer \( m \) so that \( x \in [m - 1, m) \) consider \( r = [na]/n \).

(b) \( (a, b) \cap \mathbb{Q}^c \) is non-empty (\( \mathbb{Q}^c \) is the complement of \( \mathbb{Q} \)).

**Hint.** One approach is to first show that \( \sqrt{2}r \) is irrational for every non-zero rational \( r \). (You may assume \( \sqrt{2} \) is irrational.)

**Solution.** (a) Clearly \( r = [na]/n \) is rational. Note that if \( n \) is chosen as in the Hint,
\[
a = na \frac{na}{n} < \frac{[na]}{n} \leq \frac{na+1}{n} = a + \frac{1}{n} < a + (b - a) = b,
\]
so that \( r \in (a, b) \).

(b) Let \( r \neq 0 \) be rational. If \( \sqrt{2}r = s \) is rational, then \( \sqrt{2} = sr^{-1} \) is rational, a contradiction. Therefore \( A = \{ \sqrt{2}r : r \in \mathbb{Q}, r \neq 0 \} \subseteq \mathbb{Q}^c \).

Now by (a) \( (a/\sqrt{2}, b/\sqrt{2}) \) contains a rational number \( r \). If \( r = 0 \), then this interval will contain \( 1/n \) for large enough \( n \) so that \( 1/n < b/\sqrt{2} \). Therefore by replacing \( r = 0 \) with \( 1/n \neq 0 \) we may assume \( r \neq 0 \). Therefore \( a < \sqrt{2}r < b \) and so \( (a, b) \) contains \( \sqrt{2}r \) which is irrational by the above.

6. A function \( f \) is increasing if \( x \leq x' \) implies \( f(x) \leq f(x') \), and is decreasing if \( x \leq x' \) implies \( f(x) \geq f(x') \). We say \( f \) is monotone if it is either increasing or decreasing. Let \( f : [a, b] \to \mathbb{R} \) be a decreasing function.

(a) Prove that for any partition \( P = \{x_0, x_1, \ldots, x_n\} \) of \([a, b]\),
\[
U(f, P) - L(f, P) \leq |f(b) - f(a)||P|.
\]

(Recall that \( ||P|| = \max_{i=1, \ldots, n} \Delta x_i \).)

(b) Prove that \( f \) is integrable on \([a, b]\).

(c) Use (a) and (b) to show that for any partition \( P \) of \([a, b]\),
\[
\left| \int_a^b f \, dx - L(f, P) \right| \leq |f(b) - f(a)||P| \quad \text{and} \quad \left| \int_a^b f \, dx - U(f, P) \right| \leq |f(b) - f(a)||P|,
\]
Of course similar arguments will work if \( f \) is increasing and so the above holds for any monotone \( f \).

**Solution:** (a) Let \( P = \{x_0, \ldots, x_n\} \). Since \( f \) is decreasing we have \( m_i = f(x_i) \) and \( M_i = f(x_{i-1}) \). Therefore

\[
U(f, P) - L(f, P) = \sum_{i=1}^{n} (f(x_{i-1}) - f(x_i)) \Delta x_i \\
\leq \max\{\Delta x_i : i = 1, \ldots, n\} \sum_{i=1}^{n} f(x_{i-1}) - f(x_i) \\
= \|P\| (f(b) - f(a)) = \|P\| |f(b) - f(a)|.
\]

In the last line we used the fact that we have a telescoping sum.

(b) We may assume \(|f(b) - f(a)| > 0\), since otherwise \( f \) is constant and we noted integrability in class. Let \( \epsilon > 0 \). Choose \( P \) s.t. \( \|P\| < \epsilon/|f(b) - f(a)| \). Then by (a) we have \( U(f, P) - L(f, P) \leq \|P\| |f(b) - f(a)| < \epsilon \). By the Integrability Test, \( f \) is integrable on \([a, b]\).

(c) By the definition of the Riemann integral and (b) we have,

\[
0 \leq \int_{a}^{b} f dx - L(f, P) \leq U(f, P) - L(f, P) \leq |f(b) - f(a)| \|P\|.
\]

This gives the first inequality and the second one is proved in the same way.

7. These are practice questions from the text NOT TO BE HANDED IN.

Ex. 5.1 # 12, 22, 39