Solutions to Math 121 Assignment 12

This assignment was not to be handed in.

1. Read Example 5 on p. 448 in Sec. 7.9 of text. Find the solution of the given equation with \(x(0) = 0\), but now with \(a = b > 0\).

\[
\frac{dx}{dt} = k(a - x)^2
\]
holds (until \(x(t) = a\)) iff \(\int \frac{dx}{(a-x)^2} = kt \) iff \(\frac{1}{a-x} = kt + C\) and so \(x(t) = a - \frac{1}{kt+C}\) is the general solution for all \(t \geq 0\). The initial condition \(x(0) = 0\) leads to \(C = a\) and so simplifying we get \(x(t) = a - \frac{a}{kt+1}\).

2. Solve:

\[
\frac{dy}{dx} + 3x^2y = x^2, \quad y(0) = 1.
\]

\[
\mu(x) = \int 3x^2dx = x^3
\]
so the integrating factor is \(e^{x^3}\) which leads to the general solution: \(y(x) = e^{-x^3} \int e^{x^3} x^2dx = \frac{e^{-x^3}}{3}[e^{x^3} + C'] = \frac{1}{3} + Ce^{-x^3}\). The initial condition \(y(0) = 1\) implies \(C = \frac{2}{3}\) and so \(y(x) = \frac{1}{3} + \frac{2}{3}e^{-x^3}\).

3. Solve (find the most general form of the solution):

(a) \(\frac{dY}{dt} = tY\). Ans. \(Y(t) = Ce^{t^2/2}\).

\[
\frac{dY}{dt} = tY \text{ leads to } \int \frac{dY}{Y} = \int tdt = t^2/2 \text{ and so (at least until } Y(t) = 0\text{) we have }
\log |Y| = t^2/2 + C' \text{ and so } Y = Ce^{t^2/2}. \text{ (Note that } C = 0 \text{ gives the solution 0 (which is unique but we have not shown it) and for } C \neq 0 \text{ we get solutions that never hit 0.)}
\]

(b) \(\frac{dy}{dx} = e^y \sin x\).

\[
\int e^{-y}dy = \int \sin xdx = -\cos x. \text{ Therefore } e^{-y} = \cos x + C \text{ and so } y = -\log(\cos x + C).
\]

4. Assume that the frictional force acting on an object in free fall is \(-kv^2\), where \(v = v(t)\) is the velocity of the falling object (the direction of the earth is positive). Then Newton’s second law gives: \(m\frac{dv}{dt} = mg - kv^2\). If the object is initially at rest find its velocity up until it hits the earth. If we ignore the latter event, find the limiting velocity as \(t \to \infty\).

Separating variables in the above differential equation, leads to (set \(b = k/m\)) \(\int \frac{dv}{g-bv^2} = t\) and so using partial fractions we get \(\frac{1}{2\sqrt{g}} \int \frac{1}{\sqrt{g} - \sqrt{bv}} + \frac{1}{\sqrt{g} + \sqrt{bv}} dv = t\) and hence \(\log \frac{\sqrt{g} + \sqrt{bv}}{\sqrt{g} - \sqrt{bv}} = 2\sqrt{gt} + C\). Using \(v(0) = 0\) gives \(C = 0\) and so solving for \(v\) we get \(v(t) = \frac{\sqrt{g}}{\sqrt{b}} e^{\sqrt{g/bt} - 1} \) or equivalently \(v(t) = \frac{\sqrt{g}m}{\sqrt{k}} e^{2\sqrt{g/bt} - 1/2} \).