Lecture 45 (Sec 7.9) April 3/20

1. Separable DE's

\( \frac{dy}{dx} \) (Separable Equation) \( r \in \mathbb{R} \)

\( \frac{dy}{dx} (x) = \frac{r}{1+x} \), \( x > -1 \), \( y(0) = 1 \) (need \( x > -1 \) to ensure \( \frac{1}{1+x} \) is cont'd)

\( \Rightarrow \int \frac{dy}{y} = \int \frac{r}{1+x} \) \( (x > -1) \)

\( \Rightarrow \log y = r \log (1+x) + C \)

\( \Rightarrow y(x) = e^C (1+x)^r = C_1 (1+x)^r \) \( \) (use \( y(0) = 1 \) to see \( C_1 = 1 \))

\( y(0) = 1 \), \( y(1) = (1+x)^r \), \( x > -1 \) \( \Rightarrow \) \( y(x) = (1+x)^r \) (by continuity of \( y(x) \))

So unique solution is

\( y(x) = (1+x)^r \), \( x > -1 \)

(as we could have guessed!)

\( \Rightarrow \) in above is clear and gives uniqueness.

\( \Rightarrow \) Holds by our more careful reasoning just today and gives evidence.

Of course here you check directly that

\( y(x) = (1+x)^r \) satisfies (52)!

Note also solution is valid \( \forall x > -1 \), not just \( x \geq 0 \).

Remark. Of course if \( 0 < r < 1 \) and \( y(0) \) \( \) restricted to \( \mathbb{R} \), \( y(x) \) \( \) restricted to \( \mathbb{R} \) is also the unique solution on \( I \); same argument, just restricted to \( I \).

(we'll use this on Mon.)
2. First Order Linear Differential Equations

Let \( p(x), q(x) \) be continuous on an interval \( I \) so that \( I = (a, b) \) or \( I = (a, \infty) \) or \( I = (-\infty, b) \).

We call (LDE) \( \frac{dy}{dx} + p(x)y = q(x) \)

a 1st order linear differential equation; it's linear in the solution \( y(x) \).

Note: If \( q = 0 \), we call (LDE) a homogeneous linear equation. In this case (LDE) is also separable.

\[
\frac{dy}{dx} = -\frac{y \ p(x)}{q(x)}
\]

For general \( q(x) \), it does not appear to be separable.

**Example 1**
\( p = 0 \)
\( \frac{dy}{dx} = q(x) \)
\( y = \int q(x) dx \) (LDE)

**Example 2**
\( p = \frac{1}{x} \) \( x > 1 \)
Get Binomial Equation

\[ \frac{dy}{dx} = \left( \frac{x}{x+1} \right) y \]

**Plan of attack:** Essentially reduce (LDE) by multiplying both sides of (LDE) by \( e^{\mu x} \) so where \( \mu \) is a carefully chosen \( C^1 \) function so that the LHS becomes an exact derivative: \( \frac{d}{dx} (e^{\mu x} \ y(x)) \).
\[
\frac{dy}{dx} + p(x)y = q(x) \quad (x \in I)
\]

\[
\Rightarrow 10) e^{\mu(x)} \left[ \frac{dy}{dx} + p(x)y \right] = e^{\mu(x)} q(x)
\]

We seek \( \mu(x) \) s.t.

\[
(1) \quad \int e^{\mu(x)} \left[ \frac{dy}{dx} + p(x)y \right] dx = \frac{d}{dx} \left( e^{\mu(x)} y(x) \right)
\]

We call \( e^{\mu(x)} \) an **integrating factor**.

\[
(1) \Rightarrow e^{\mu(x)} \left[ y' + p(x)y \right] = e^{\mu(x)} \left[ y' + \mu(x) y \right] \quad \text{(product rule)}
\]

\[
\Rightarrow e^{\mu(x)} p(x)y = e^{\mu(x)} \mu'(x) y(x)
\]

\[
\Rightarrow p(x)y = \mu'(x) y \quad \text{vers.}
\]

So set \( \mu(x) = \int p(x) dx \); any antiderivative will do.

It is now easy to solve 10\( x \in I \).

\[
(10) \Rightarrow 11) \quad \frac{d}{dx} \left( e^{\mu(x)} y(x) \right) = e^{\mu(x)} q(x)
\]

\[
\Rightarrow e^{\mu(x)} y(x) = \int e^{\mu(x)} q(x) dx + c
\]

In 13) we use the general antiderivative (ie include \( c \))

to accommodate "IC's" like \( y(x_0) = y_0 \).

We have proved a theorem...
Thm 29.1 Assume \( p, q \) are continuous functions on an interval \( I \) and \( \mu(x) = p(x) \) on \( I \). \( \mu \) is a particular antiderivative (i.e., \( \frac{d}{dx} \mu = \mu \)). Then, \( y(x) \) solves

\[
(\begin{align*}
(1) & \ y' + p(x) y = q(x) \\
(2) & \ y(0) = 1
\end{align*})
\]

whence

\[
y(x) = e^{-\int_0^x p(t) \, dt} q(t) \, dt + C_1
\]

Proof. Let

\[
y' - xy = 1
\]

be exact. Then, by integrating factor and linearity,

\[
\int_0^x e^{-\frac{x^2}{2}} (y' - xy) \, dx = e^{-\frac{x^2}{2}}
\]

\[
\frac{1}{2} \int_0^x (e^{-\frac{x^2}{2}} y) \, dx = e^{-\frac{x^2}{2}}.
\]

\[
e^{-\frac{x^2}{2}} y = \int_0^x e^{-\frac{x^2}{2}} \, dx = \int_0^\infty e^{-\frac{u^2}{2}} \, du + C.
\]

For picked \( \int_0^\infty \), 1 of \( y(0) \) is specified.

\[
x = 0 \quad y(0) = 1 \Leftrightarrow 1 = 0 + C \quad \Leftrightarrow \quad C = 1
\]

so

\[
y = e^{\frac{x^2}{2}} \left( \int_0^x e^{-\frac{u^2}{2}} \, du + e^{\frac{x^2}{2}} \right)
\]

\( \forall x \in I \).
E.g. Assume \( v(t) \) = velocity of mass \( m \) rolling to earth.

\[ v(t) = v_0 e^{-at} \]

Assume force of air resistance is proportional to \( v \). Find \( v(t) \).

\[ \begin{align*}
\vec{F}_{\text{grav}} &= -g \hat{m} \\
\vec{F}_{\text{air res}} &= -k v(t) \\
\vec{F} &= (g m - k v(t)) \\
\end{align*} \]

In x-direction. Newton's 2nd law.

\[ \begin{align*}
m a(t) &= m v'(t) \\
\end{align*} \]

gives

\[ g m - k v = m v' \]

\[ (v' + \frac{k}{m} v(t)) = g \]

\[ (v' + \frac{k}{m} v) = \frac{g}{m} \]

so \( v(t) = \frac{k}{m} \int \frac{v(t)}{m} dt \) is linear and separable.

\[ a(t) = \frac{k}{m} v(t) \]

\[ e^{\int a(t) dt} \cdot v(t) = e^{\int \frac{k}{m} v(t) dt} \]

\[ e^{\frac{k}{m} t} \cdot (v + \frac{k}{m} V) = e^{\frac{k}{m} t} \cdot g \]

\[ \int e^{\frac{k}{m} t} v dt = \int g e^{\frac{k}{m} t} dt \]

\[ e^{\frac{k}{m} t} \cdot \int e^{\frac{k}{m} t} v dt = \int g e^{\frac{k}{m} t} dt \]

\[ \int e^{\frac{k}{m} t} v dt = \frac{g}{\frac{k}{m}} \int e^{\frac{k}{m} t} dt + C \]

\[ \int e^{\frac{k}{m} t} v dt = \frac{g}{\frac{k}{m}} e^{\frac{k}{m} t} + C \]

\[ v(0) = \frac{g}{d} + C \Rightarrow C = \frac{g}{d} - \frac{g}{a} \]

\[ v(t) = \frac{g}{d} + C e^{-\frac{k}{m} t} = \frac{g}{d} + \left(\frac{g}{d} \right) e^{-\frac{k}{m} t} \]

\[ v(t) = \frac{g m}{k} + \left(\frac{v_0 - \frac{g m}{k}}{e^{-\frac{k}{m} t}} \right) e^{-\frac{k}{m} t} \]

\[ \lim_{t \to \infty} \frac{g m}{k} \]

\[ \text{Terminal velocity.} \]
More often one assumes $\mathbf{F}_{air} = -k v^2 \mathbf{k}$

Leading to:

$$v'(0) + \frac{k}{m} v^2(0) = 0, \quad v(0) = v_0$$

or $$v'(0) = g - \frac{k}{m} v^2(0), \quad v(0) = v_0$$

This is no longer linear but is still separable \( v' \mathbf{k} = h(\mathbf{v}) \)

**HELP!** Find \( v \) and terminal velocity.

\[
v = v_{lim} \sqrt{\frac{mg}{k}} \quad (\text{As } v \to v_{lim} \text{, } v' \to 0 \text{ so easy to find})
\]
A Centre of Mass Example

(Practice Q. Huij, En5 Sec 7.3)

A wire is traced out by the 1-1 C' parametric curve \( r(t) = (x(t), y(t)) \)

The density of the wire at \((x,y)\) is \( \rho(x,y) > 0 \). (Problem)

(a) Show its mass \( m = \int \rho(x,y) \, ds \)

(b) Show its co-m. \( C(x,y) = \left( \frac{M_x}{m}, \frac{M_y}{m} \right) \), where

\[
M_x = \int x \rho(x,y) \, ds, \\
M_y = \int y \rho(x,y) \, ds,
\]

we approximate partitioned wire

(a) Let \( S_i \) be \( \frac{1}{i} \) part of \( \frac{1}{i-1} \). We approximate \( S_i \) by the line segment (from \( a_ib_i \) to \( a_{i+1}b_{i+1} \)) with constant density \( \rho(a_ib_i) \).

\[
S_i = \text{mass of } S_i \approx \text{mass of } a_{i+1}b_{i+1} = \rho(a_{i+1}b_{i+1}) \, \parallel (a_{i+1}b_{i+1}) - (a_ib_i) \parallel \, ds_{i+1}.
\]

Similarly \( m = \sum_{i=1}^{n} \frac{S_i}{n} \),

\[
M_x = \sum_{i=1}^{n} \frac{\rho(a_{i+1}b_{i+1}) \, ds_{i+1}}{n} \, \parallel (a_{i+1}b_{i+1}) - (a_ib_i) \parallel \, ds_{i+1},
\]

(b) Let \( M_x^1 = x \text{-moment of } S_i \approx \sum_{i=1}^{n} \frac{\rho(a_{i+1}b_{i+1}) \, ds_{i+1}}{n} \, \parallel (a_{i+1}b_{i+1}) - (a_ib_i) \parallel \, ds_{i+1}, \)

\[
M_x = \sum_{i=1}^{n} \frac{\rho(a_{i+1}b_{i+1}) \, ds_{i+1}}{n} \, \parallel (a_{i+1}b_{i+1}) - (a_ib_i) \parallel \, ds_{i+1},
\]

Similarly \( M_y = \sum_{i=1}^{n} \frac{\rho(a_{i+1}b_{i+1}) \, ds_{i+1}}{n} \, \parallel (a_{i+1}b_{i+1}) - (a_ib_i) \parallel \, ds_{i+1} \).
\[ \bar{x} = \frac{M_x}{M} = \frac{\int_0^{\pi/2} r \cos \theta \, d\theta}{\int_0^{\pi/2} r \, d\theta} = \frac{r_0^2 \int_0^{\pi/2} \cos \theta \, d\theta}{r_0 \int_0^{\pi/2} \, d\theta} \]

By symmetry, \( \bar{y} = \bar{x} = \frac{2}{\pi} r_0 \)