I wrote the following 7 questions not only with the motivation of providing you with some practice for your final exam, but also with the intent of exposing you to many ideas that are fundamental to a lot of higher-level mathematics. These problems can be fairly difficult, but I believe that they are within reach given what we have learned in this course. I intentionally made them more challenging as I know you have access to your textbook and WebWork for the more elementary problems. I also tried to provide you with some motivation in my remarks following each problem. Happy studying everybody!

1: Let $p > 1$ and let $q > 1$ be the unique solution to
\[ \frac{1}{p} + \frac{1}{q} = 1. \]
We call $q$ the “conjugate exponent” to $p$. Suppose $f, g \geq 0$ are continuous functions defined on all of $\mathbb{R}$.

(a): For $u, v \geq 0$, prove that
\[ uv \leq \frac{u^p}{p} + \frac{v^q}{q}. \]
Furthermore, prove that equality holds if and only if $u^p = v^q$.

(b): Assume that
\[ \int_a^b f^p \, dx < \infty \quad \text{and} \quad \int_a^b g^q \, dx < \infty. \]
Prove that
\[ \int_a^b fg \, dx \leq \left( \int_a^b f^p \, dx \right)^{1/p} \left( \int_a^b g^q \, dx \right)^{1/q}. \]
*Hint:* First prove this is true if the expressions in (1) are equal to 1.

(c): Extend the result in part (a) to improper integrals. That is, if
\[ \int_0^\infty f^p \, dx < \infty \quad \text{and} \quad \int_0^\infty g^q \, dx < \infty. \]
Prove that
\[ \int_0^\infty fg \, dx \leq \left( \int_0^\infty f^p \, dx \right)^{1/p} \left( \int_0^\infty g^q \, dx \right)^{1/q}. \]
*Remark:* This is the famous Hölder’s inequality. It is of significant importance to most areas of analysis. The special case of $p = q = 2$ is a special case of the Cauchy Schwarz inequality which is (arguably) of even greater importance.

2(a): Suppose $1 < s < \infty$. Define
\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \]
Prove that
\[ \zeta(s) = s \int_1^\infty \frac{x}{x^s+1} \, dx. \]
Here $\lfloor x \rfloor$ denotes the greatest integer $\leq x$. *Hint:* One way to solve this is to first show that
\[ \int_1^\infty \frac{x}{x^s+1} \, dx = \sum_{n=1}^{\infty} \int_n^{n+1} \frac{n}{x^s+1} \, dx. \]
You should be quite careful in the treatments of the limits but from here, you should be able to manipulate the sum to what you want.
(b): Furthermore, prove that
\[\zeta(s) = \frac{s}{s-1} - s \int_{1}^{\infty} \frac{x - \lfloor x \rfloor}{x^{s+1}} \, dx.\]

(c): Prove that the integral in (b) converges for all \(s > 0\).

Remark: This is the Riemann Zeta function. Of extreme importance in number theory and complex analysis. The famous unsolved Riemann Hypothesis is concerning the zeros of an extension of \(\zeta\).

3: First, a definition. We say a sequence \(\{x_n\}_{n=1}^{\infty}\) of real numbers is Cauchy if for all \(\epsilon > 0\), there exists \(N > 0\) such that for all \(n, m \geq N\), \(|x_m - x_n| < \epsilon\). We say \(\{x_{n_k}\}_{k=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty}\) is a subsequence if \(k_1 < k_2\) implies \(n_{k_1} < n_{k_2}\). For example, a subsequence of \((1, 2, 3, 4, 5, \ldots)\) is \((2, 5, 9, 1000, \ldots)\).

(a): Prove that a convergent sequence is Cauchy.

(b): Prove that every Cauchy sequence is bounded. Conclude that in particular, every convergent sequence is bounded.

(c): Prove that if a sequence \(\{x_n\}_{n=1}^{\infty}\) is Cauchy and some subsequence \(\{x_{n_k}\}_{k=1}^{\infty}\) converges, then \(\{x_n\}_{n=1}^{\infty}\) converges and the limit is the same. That is, if \(\lim_{k \to \infty} x_{n_k} = a\), then \(\lim_{n \to \infty} x_n = a\).

(d): Construct a Cauchy sequence \(\{x_n\}_{n=1}^{\infty}\) such that \(x_n \in \mathbb{Q}\) for all \(n \in \mathbb{N}\) and converges to some \(x \in \mathbb{R} \setminus \mathbb{Q}\). Note that you need to prove 3 things: (1) that your series is in fact Cauchy; (2) that the sequence converges to some point \(x\); (3) that \(x \notin \mathbb{Q}\).

Remark: It turns out that every Cauchy sequence of real numbers converges to some real number. Part (c) however illustrates an important point: The real numbers are much richer than just the rationals. In fact, we call \(\mathbb{R}\) the completion of \(\mathbb{Q}\) for this very reason.

4: This problem is courtesy of Professor Malabika Pramanik; A particularly excellent teacher and researcher.

(a): Show for any \(n \in \mathbb{N}\),
\[\int_{-\pi}^{\pi} \cos(nx) \, dx = \int_{-\pi}^{\pi} \sin(nx) \, dx = 0.\]

(b): For \(m, n \in \mathbb{Z}\), compute the following:
\[\int_{-\pi}^{\pi} \cos(mx) \cos(nx) \, dx, \quad \int_{-\pi}^{\pi} \sin(mx) \sin(nx) \, dx, \quad \int_{-\pi}^{\pi} \sin(mx) \cos(nx) \, dx.\]

(c): Now suppose for some \(N \in \mathbb{N}\),
\[f(x) = \frac{a_0}{2} + \sum_{n=1}^{N} (a_n \cos(nx) + b_n \sin(nx)) \text{ for all } x \in [-\pi, \pi].\]

Solve for \(a_n, b_n, a_0\) via some integrals involving \(f\). Hint: To find \(a_0\), use part (a). To find the rest of the coefficients, multiply by an appropriate choice of function, integrate, then use part (b).

Remark: The coefficients \(a_n, b_n, a_0\) are called the Fourier coefficients of \(f\). Fourier coefficients arise in a variety of contexts, such as communications and signal processing. If \(f\) is a musical note, then the integers \(n\) for which \(a_n\) or \(b_n\) are nonzero are precisely the frequencies comprising the note.

5: For the remainder of this problem, \(a < b\) and \(f : [a, b] \to \mathbb{R}\). Note that we proved results (b) and (c) in Math 120, but this was possibly the most challenging problem in that course. It would do you all
good to review this material.

(a): Prove that if $f$ is discontinuous only at finitely many points in $[a, b]$, then $f$ is integrable on this interval.

(b): Recall a very special function from Math 120:

$$g(x) = \begin{cases} \frac{1}{q} & : x = \frac{p}{q} \text{ (in lowest form)} \\ 0 & : x \notin \mathbb{Q} \end{cases}$$

Prove that $g$ is discontinuous at every $x \in [a, b] \cap \mathbb{Q}$.

(c): Prove that $g$ is continuous as every $x \in [a, b] \cap (\mathbb{R} \setminus \mathbb{Q})$.

(d): Prove that $g$ is integrable on $[a, b]$. *Hint:* You somehow need to control the value of $g$ on the rationals. Try to simplify your notation by first assuming $a = 0, b = 1$. Based on $\epsilon$, you should be able to create a partition to pass the integrability test.

(e): In the definition of $g$, replace $1/q$ simply by 1. Prove that this new $g$ is not integrable.

*Remark:* This problem aims to show you multiple things: (1) a function can be discontinuous at infinitely many points but still be integrable, (2) for Riemann integration, there is a huge difference between functions that are defined very similarly.

6(a): If $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and $a < b$, use IBP to prove

$$\lim_{r \to \infty} \int_a^b f(x) \sin(rx) dx = 0$$

Note that you should be very careful in your treatment of limits.

(b): If $a_n$ decreases to 0 (that is, $a_{n+1} \leq a_n$ and $a_n \to 0$) and $\sum_{n=1}^{\infty} a_n$ converges, prove that $\sum_{n=1}^{\infty} 2^n a_{2^n}$ also converges. Use this to prove that the harmonic series diverges.

(c): Let $\{a_n\}_{n=1}^{\infty}$ be a sequence such that $\sum_{n=1}^{\infty} |a_n|$ converges (that is, it converges absolutely). Then for any $n_1 < n_2 < n_3 < \ldots$, prove that $\sum_{j=1}^{\infty} a_{n_j}$ converges. That is, if a series is absolutely convergent, then any subseries converges.

(d): Now suppose that $\sum_{n=1}^{\infty} a_n$ converges conditionally. Prove that there exists $n_1, n_2, \ldots$ such that $\sum_{j=1}^{\infty} a_{n_j}$ diverges.

7(a): Suppose that for each $n \in \mathbb{N}$, $m \in \mathbb{N}$, $a_{n,m} \geq 0$. Assume

$$\sup_{N,M \in \mathbb{N}} \sum_{n \leq N, m \leq M} a_{n,m} < \infty.$$ 

Prove that

$$\sum_{n=1}^{\infty} a_{n,m} < \infty \text{ for all } m \in \mathbb{N} \text{ and } \sum_{m=1}^{\infty} a_{n,m} < \infty \text{ for all } n \in \mathbb{N}$$

(b): Under the assumption of part (a), prove that

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{n,m} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{n,m} = \sup_{N,M \in \mathbb{N}} \sum_{n \leq N, m \leq M} a_{n,m}$$

by following this prescribed outline.
(i) First observe that the problem is symmetric in $n$ and $m$ so argue that it suffices to prove

$$
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{n,m} = \sup_{N,M \in \mathbb{N}} \sum_{n \leq N, m \leq M} a_{n,m}.
$$

(ii) When you want to prove equalities like this, it is usually best to prove that

$$
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{n,m} \leq \sup_{N,M \in \mathbb{N}} \sum_{n \leq N, m \leq M} a_{n,m}, \quad \text{and} \quad \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{n,m} \geq \sup_{N,M \in \mathbb{N}} \sum_{n \leq N, m \leq M} a_{n,m}.
$$

Argue that the “$\geq$” case is always true.

(iii) The “$\leq$” case is much harder. Start by letting $\epsilon > 0$. Try to prove that

$$
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{n,m} \leq \sup_{N,M \in \mathbb{N}} \sum_{n \leq N, m \leq M} a_{n,m} + \epsilon.
$$

This would complete the proof (why?).

Remark: It is often the case that one would want to re-arrange the order of limits. This problem can be generalized fairly easily to give you one way that is valid. The technique you use to prove this problem is a very good one to be familiar with.